1

MATRICES

UNIT STRUCTURE

1.0 OBJECTIVES

In this chapter a student has to learn the

- Concept of adjoint of a matrix.
- Inverse of a matrix.
- Rank of a matrix and methods finding these.

1.1 INTRODUCTION

At higher secondary level, we have studied the definition of a matrix, operations on the matrices, types of matrices inverse of a matrix etc.

In this chapter, we are studying adjoint method of finding the inverse of a square matrix and also the rank of a matrix.

1.2 DEFINITIONS

1) **Definitions:-** A system of $m \times n$ numbers arranged in the form of an ordered set of m horizontal lines called rows & n vertical lines called columns is called an $m \times n$ matrix.

The matrix of order $m \times n$ is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2j} & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} & a_{i3} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & a_{mj} & a_{mn} \end{bmatrix}_{n\times n}$$

Note:

i) Matrices are generally denoted by capital letters.

ii) The elements are generally denoted by corresponding small letters.

Types of Matrices:

1) Rectangular matrix :-

Any mxn Matrix where $m \neq n$ is called rectangular matrix.

For e.g

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix}_{2\times 3}$$

2) Column Matrix :

It is a matrix in which there is only one column.

$$x = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}$$

3) Row Matrix :

It is a matrix in which there is only one row.

$$x = \begin{bmatrix} 5 & 7 & 9 \end{bmatrix}_{1\times 3}$$

4) Square Matrix :

It is a matrix in which number of rows equals the number of columns.

i.e its order is n x n.

e.g.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}_{2\times 2}$$

5) Diagonal Matrix:

It is a square matrix in which all non-diagonal elements are zero.

e.g.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

6) Scalar Matrix:

It is a square diagonal matrix in which all diagonal elements are equal.

e.g.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$$

7) Unit Matrix:

It is a scalar matrix with diagonal elements as unity.

e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

8) Upper Triangular Matrix:

It is a square matrix in which all the elements below the principle diagonal are zero.

e.g.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$$

9) Lower Triangular Matrix:

It is a square matrix in which all the elements above the principle diagonal are zero.

e.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}_{3\times 3}$$

10) Transpose of Matrix:

It is a matrix obtained by interchanging rows into columns or columns into rows.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 9 \end{bmatrix}_{2\times 3}$$

$$A^T = Transpose\ of\ \mathrm{A} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \\ 5 & 9 \end{bmatrix}_{3\times 2}$$

11) Symmetric Matrix:

If for a square matrix A, $A = \mathrm{A}^T$ then A is symmetric

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 1 \\ 5 & 1 & 9 \end{bmatrix}$$

12) Skew Symmeric Matrix :

If for a square matrix A, $A = \mathrm{A}^T$ then it is skew -symmetric matrix.

$$A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix}$$

Note : For a skew Symmetric matrix, diagonal elements are zero.

Determinant of a Matrix:

Let A be a square matrix then

$| A | = determinant\ of\ A\ \ i.e\ det\ A = |A|$

If (i) then $|A| \neq 0$ matrix A is called as non-singular and
If (i) then $|A| = 0,$ matrix A is singular.

Note : for non-singular matrix A-1 exists.

a) Minor of an element :

Consider a square matrix A of order n

Let
$A = \left[a_{ij} \right]_{n \times n}$

The matrix is also can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & - & - & - & a_{nn} \end{bmatrix}$$

Minor of an element a_{ij} is a determinant of order (nd) by deleting the elements of the matrix A, which are in 6th row and 5th column of A.

E.g. Consider,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{11} = Minor of an element a_{11}

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

II y

$$M_{12} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix}$$

E.g.

(ii) Let,

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}, M_{13} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 2 & 8 \\ 0 & 6 \end{bmatrix}, M_{23} = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

(b) Cofactor of an element :-

If $A = \left[a_{ij}\right]$ is a square matrix of order n and a_{ij} denotes cofactor of the element a_{ij}.

$C_{ij} = (-1)^{i+j} . M_{ij}$ Where M_{ij} is minor of a_{ij}.

$$\text{If } A = \begin{bmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$$

A_1 = The cofactor of $A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

B_1 = The cofactor of $b_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

C_1 = The cofactor of $b_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

E.g. Consider,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$$

$$c_{11} = (-1)^{1+1} M_{11} \quad c_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix}$$

$$= (-1)^{1+1} . \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \quad = (-1)^{3} \times (0-3)$$

$$= (-1)^{1+1} . \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \quad = (-1)^{3} \times (0-3)$$

$$= (1) \times (12-7) \quad = (-1) \times (-3)$$

$$= (1) \times (12-7) \quad = (-1) \times (-3)$$

$$= 5 \quad = 3$$

(C) Cofactor Matrix :-

A matrix C = $[C_{ij}]$ where C_{ij} denotes cofactor of the element a_{ij}.
Of a matrix A of order nxn, is called a cofactor matrix.

In above matrix A, cofactor matrix is

$$C = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Similarly for a matrix, A = $\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ the cofactor matrix is c= $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(d) Adjoint of Matrix :-

If A is any square matrix then transpose of its cofactor matrix is called Adjoint of A.

Thus in the notations used,

Adjoint of $A = C^T$

$$\Rightarrow Adj\ A = \begin{bmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Adjoint of a matrix A is denoted as Adj.A

Thus if,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix} \text{ than Adj. } A = \begin{bmatrix} 5 & -10 & 3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

Note :

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ than Adj . $A = \begin{bmatrix} \mathrm{d} & -b \\ -c & a \end{bmatrix}$

(d) Inverse of a square Matrix:-

Two non-singular square matrices of order n A and B are said to be inverse of each other if,

AB=BA=I, where I is an identity matrix of order n.

Inverse of A is denoted as A^{-1} and read as A inverse.

Thus
$AA^{-1}=A^{-1}A=I$

Inverse of a matrix can also be calculated by the Formula.

$A^{-1} = \frac{1}{|A|}$ Adj.A where $|A|$ denotes determinant of A.

Note:- From this relation it is clear that A^{-1} exist if and only if $|A| \neq 0$ i.e A is non singular matrix.

1.3 ILLUSTRATIVE EXAMPLES

Example 1: Find the inverse of the matrix by finding its adjoint

$$A=\begin{bmatrix}2 & 1 & 3\\ 3 & 1 & 2\\ 1 & 2 & 3\end{bmatrix}$$

Solution: We have,

$$|A| = 2\ (3-4) - 1\ (9-2) + 3\ (6-1)$$
$$= -2-7+15$$
$$|A| = 6$$
$$|A| \neq 0$$
A^{-1} exists

Transpose of matrix A=A^1

$$\therefore \quad A^1 = \begin{bmatrix}2 & 3 & 1\\ 1 & 1 & 2\\ 3 & 2 & 3\end{bmatrix}$$

We find co-factors of the elements of A^1 (Row-wise)

$$C.F.(2) = -1, \quad C.F.(3) = 3, \quad C.F.(1) = -1$$
$$\therefore \quad C.F.(1) = -7, \quad C.F.(1) = 3, \quad C.F.(2) = -5$$
$$C.F.(3) = 5, \quad C.F.(2) = -3, \quad C.F.(3) = -1$$

$$\therefore \quad \text{adj } (A) = \begin{bmatrix}-1 & 3 & -1\\ -7 & 3 & -5\\ 5 & -3 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj (A)} = \frac{1}{6}\begin{bmatrix}-1 & 3 & -1\\ -7 & 3 & -5\\ 5 & -3 & -1\end{bmatrix}$$

Example 2: Find the inverse of matrix A by Adjoint method, if

$$A = \begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$$

Solution: Consider

$$|A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= 0(-1) - 1(-8) + 2(-5)$$
$$= 0 + 8 - 10$$
$$= -2$$

Co factor of the elements of A are as follows

$$C_{11} = (-1)^{1+1} . \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2} . \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$$

$$C_{13} = (-1)^{1+3} . \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{21} = (-1)^{2+1} . \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2} . \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$$

$$C_{23} = (-1)^{2+3} . \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$$

$$C_{31} = (-1)^{3+1} . \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{32} = (-1)^{3+2} . \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3} . \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

Thus,

Cofactor of matrix $C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

And Adjoint of A= C^1

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix} \Rightarrow A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

Note:- A Rectangular matrix does not process inverse.

Properties of Inverse of Matrix:-

i) The inverse of a matrix is unique i.e

ii) The inverse of the transpose of a matrix is the transpose of inverse i.e. $(A^T)^{-1} = (A^{-1})^T$

iii) If A & B are two non-singular matrices of the same order $(AB)^{-1} = B^{-1}A^{-1}$

This property is called reversal law.

Definition:-Orthogonal matrix.:-

If a square matrix it satisfies the relation $AA^T = I$ then the matrix A is called an orthogonal matrix. &

$A^T = A^{-1}$

Example 3:

show that $A = \begin{bmatrix} Cos\theta & Cos\theta \\ Sin\theta & Cos\theta \end{bmatrix}$ is orthogonal matrix.

Solution:

To show that A is orthogonal i.e To show that $AA^T = I$

$$A = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}\begin{bmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$= \begin{bmatrix} \text{Cos}^2\theta + \text{Sin}^2\theta & -\text{Cos}\theta\text{Sin}\theta + \text{Sin}\theta\text{Cos}\theta \\ -\text{Sin}\,\theta\text{Cos}\theta + \text{Cos}\theta\text{Sin}\theta & \text{Sin}^2\theta + \text{Cos}^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is an orthogonal matrix.

Check Your Progress:

Q. 1) Find the inverse of the following matrices using Adjoint method, if they exist.

i) $\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}$, ii) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$, iii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$,

iv) $\begin{vmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{vmatrix}$, v) $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$, vi) $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{vmatrix}$

vii) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix}$

Q.3) If A = $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$, B = $\begin{vmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{vmatrix}$, C= $\begin{vmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{vmatrix}$,,

prove that $A = B.C^{-1}$

Q. 4) If A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, prove that Adj. A= A

Q. 5) If A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, verify if $(Adj.A)^1 = (Adj.A^1)$

Q.6) Find the inverse of A = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$, hence find inverse of

A = $\begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 6 & 6 & 9 \end{bmatrix}$

1.4 RANK OF A MATRIX

a) Minor of a matrix

Let A be any given matrix of order mxn. The determinant of any submatrix of a square order is called minor of the matrix A.

We observe that, if ‘r’ denotes the order of a minor of a matrix of order mxn then $1 \le r \le m$ if m<n and $1 \le r \le n$ if n<m.

e.g. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 4 & 0 & 1 & 7 \\ 8 & 5 & 4 & -3 \end{bmatrix}$$

The determinants

$$\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ 8 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 7 \\ 5 & 4 & -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 4 \\ 4 & 1 & 7 \\ 8 & 4 & -3 \end{bmatrix},$$

$$\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 5 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 0 & 7 \end{vmatrix}, |1|, |0|, |-3|,$$

Are some examples of minors of A.

b) Definition – Rank of a matrix:

A number ‘r’ is called rank of a matrix of order mxn if there is almost one minor of the matrix which is of order r whose value is non-zero and all the minors of order greater than ‘r’ will be zero.

e.g.(i) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Consider e.g. Let

$$A_1 = \begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4,\ A_2 = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \text{ etc.}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix} = 1(23) + 2(-2) = 19 \neq 0$$

$\therefore$ Rank of A= 3

(ii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Here,

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = 1(1) - 1(-1) + 2(-1) = 0$$

$$A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$$

Thus minor of order 3 is zero and atleast one minor of order 2 is non-zero $\therefore$ Rank of A = 2.

Some results:

(i) Rank of null matrix is always zero.

(ii) Rank of any non-zero matrix is always greater than or equal to 1.

(iii) If A is any mxn non-zero matrix then Rank of A is always equal to rank of A.

(iv) Rank of transpose of matrix A is always equal to rank of A.

(v) Rank of product of two matrices cannot exceed the rank of both of the matrices.

(vi) Rank of a matrix remains unleasted by **elementary transformations**.

Elementary Transformations:

Following changes made in the elements of any matrix are called elementary transactions.

(i) Interchanging any two rows (or columns) .

(ii) Multiplying all the elements of any row (or column) by a non-zero real number.

(iii) Adding non-zero scalar multitudes of all the elements of any row (or columns) into the corresponding elements of any another row (or column).

Definition:- Equivalent Matrix:

Two matrices A & B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order & the same rank. It can be denoted by

[it can be read as A equivalent to B]

Example 4: Determine the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution:

$$\text{Given } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$R_2 \Rightarrow R_2 - R_1 \quad \& \quad R_3 \Rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Here two column are Identical . hence 3rd order minor of A vanished Hence 2nd order minor $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$

$\therefore e(A) = 2$

Hence the rank of the given matrix is 2.

1.5 CANONICAL FORM OR NORMAL FORM

If a matrix A of order mxn is reduced to the form $\begin{bmatrix} I_r & o \\ o & o \end{bmatrix}$ using a sequence of elementary transformations then it called canonical or normal form. Ir denotes identity matrix of order ‘r’ .

Note:-

If any given matrix of order mxn can be reduced to the canonical form which includes an identity matrix of order ‘r’ then the matrix is of rank ‘r’.

e.g. (1) Consider

Example 5: Determine rank of the matrix. A if

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_2 - 7R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 33 & 66 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_1 - R_2,\ R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 28 & -56 \end{bmatrix}$$

$R_3 \times \frac{1}{28}$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 +\ 32\,R_3,\ \ R_2 -\ 33\,R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sim \begin{bmatrix} I_3 & o \end{bmatrix}$

$\therefore$ Rank of A=3

Example 6: Determine the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sim \begin{bmatrix} I_2 & 0 \end{bmatrix}$

$\therefore$ Rank of A= 2

Example 7: Determine the rank of matrix A if

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R_1,$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 - R_3$

$$\sim\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_1 + R_2,\ R_3 - 4R_2,\ R_4 - 9R_2$

$$\sim\begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 - 2R_3$

$$\sim\begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \times \dfrac{1}{11}$

$$\sim\begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - C_4$

$$\sim\begin{bmatrix} 1 & 0 & -1 & -7 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 + R_3,\ R_2 + 3R_3$

$$\sim\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 - \left(5C_1 + 3C_2 + 2C_2\right)$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of A= 3

Check Your Progress:-

Reduce the following to normal form and hence find the ranks of the matrices.

i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$ iii) $\begin{vmatrix} -3 & 4 & 6 \\ 5 & -5 & 7 \\ 3 & 1 & -4 \end{vmatrix}$

iv) $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{vmatrix}$ v) $\begin{vmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$ vi) $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{vmatrix}$

vii) $\begin{vmatrix} 2 & 6 & -2 & 6 & 10 \\ -3 & 3 & -3 & -3 & -3 \\ 1 & -2 & 4 & 3 & 5 \\ 2 & 0 & 4 & 6 & 10 \\ 1 & 0 & 2 & 3 & 5 \end{vmatrix}$ viii) $\begin{vmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{vmatrix}$

1.6 NORMAL FORM PAQ

If A is any mxn matrix 'r' then there exist non singular matrices P and Q such that,

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

We observe that, the matrix A can be expressed as

A = Im In …………(i)

Where Im In are the identity matrices of order m and n respectively. Applying the elementary transformations on this equation. A in L.H.S. can be reduced to normal form. The equation can be transformal into the equations.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ \ldots\ldots\ldots\ldots(ii)$$

Note that, the row operations can be performed simultaneously on L.H.S. and prefactor (i.e. Im in equation (i)) and column operations can be performed simultaneously on L.H.S. and post factor in R.H.S. i.e. [(In in eqn (i)]

Examples 8: Find the non-singular matrices P and Q such that PAQ is in normal and hence find the rank of A.

i) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix}$

Solution: Consider

A= I3 AI 3

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 5 & -4 \\ 3 & 4 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 5C$, $C_3 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -11 & -11 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1$, $R_3 - 2R_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -11 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 + C_2$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&-11&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&1&-1\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-5&-1\\0&1&1\\0&0&1\end{bmatrix}$$

$R_3\times\frac{1}{11}$,

$$\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&1&-1\\1/11&0&-2/11\end{bmatrix}A\begin{bmatrix}1&-5&-1\\0&1&1\\0&0&1\end{bmatrix}$$

$R_2\leftrightarrow R_3$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\-1/11&0&2/11\\-1&1&-1\end{bmatrix}A\begin{bmatrix}1&-5&-1\\0&1&1\\0&0&1\end{bmatrix}$$

Thus

$$P=\begin{bmatrix}0&0&1\\-1/11&0&2/11\\-1&1&-1\end{bmatrix}\ \Delta\ |P|=\frac{-1}{11}$$

$$Q=\begin{bmatrix}1&-5&-1\\0&1&1\\0&0&1\end{bmatrix}\qquad \Delta\ |Q|=1$$

P and Q are non-singular matrices
Also Rank of A = 2

ii) $\quad A=\begin{bmatrix}2&1&-3&-6\\3&-3&1&2\\1&1&1&2\end{bmatrix}$

Solutions:

Consider

$$A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1,\ C_3 - C_1,\ C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 6R_3,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 56 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times \frac{1}{28},\ R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3/14 & 1/28 & 9/28 \\ -1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3 \quad 0] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix}, \ |P| = \frac{1}{28}$$

$$Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, |Q| = 1$$

$\therefore$ P&Q are non singular.

Also,

Rank of A = 3.

Check Your Progress:

A) Find the non-singular matrices P and Q such that PAQ is in normal form and hence find rank of matrix A.

i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 3 & 4 & 7 \\ -3 & 4 & 7 & -9 \\ 5 & 4 & 6 & -5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$

1.7 LET US SUM UP

- Definition of matrix & its types.
- Using Adjoint method to find the A^{-1} by using formula $A^{-1} = \frac{1}{|A|} adjA$
- Rank of the matrix using row & column transformation
- Using canonical & normal form to find Rank of matrix.

1.8 UNIT END EXERCISE

1) Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ if exists.

ii) Find Adjoint of Matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

iii) Find the inverse of A by adjoint method if $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 2 & 3 & 1 & 0 \end{bmatrix}$

iv) Find Rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

v) Prove that the matrix $A = \begin{bmatrix} Cos\theta & -Sin\theta & 0 \\ Sin\theta & Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is orthogonal

Also find A^{-1}.

vi) Reduce the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ to the normal form θ and find its rank.

vii) Find the non singular matrix ρ *and* α. such that $\rho A \alpha$ is the normal form when $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Also find the rank of matrix B

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \& \text{ Y} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

viii) Under what condition the rank of the matrix will be 3!

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & \lambda \end{bmatrix}$$

ix) If $X = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ & $Y = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Then show that $\rho(xy) \neq \rho(yx)$ where ρ denotes Rank.

x) Find the rank of matrix $A = \begin{bmatrix} 8 & 3 & 6 & 1 \\ -1 & 6 & 4 & 2 \\ 7 & 9 & 10 & 3 \\ 15 & 12 & 16 & 4 \end{bmatrix}$

*****

2

LINEAR ALGEBRIC EQUATIONS

UNIT STRUCTURE

2.1 OBJECTIVES

After going through this chapter you will be able to

- Find the rank of Matrix.
- Find solution for linear equations.
- Type of linear equations.
- Find solution for Homogeneous equations.
- Find solution of non-Homogeneous equations.

2.2 INTRODUCTION

In XIIth we have solved linear equations by using method of reduction also by rule. Here we are going to find solution of homogeneous and non-homogeneous both with different case. Using matrix we can discuss consistency of system of equation.

2.3 CACONICAL OR ECHOLON FORM OF MATRIX

Let A be a given matrix. Then the canonical or Echelon form of A is a matrix in which

(i) One or more elements in each of first r-rows are non-zero and these first r-rows form an upper triangular matrix.

(ii) The elements in the remaining rows are zero.

Note :

1) The number of non-zero rows in Echelon form is the rank of the matrix.

2) To reduce the matrix to Echelon form only row transformations are to be applied.

Solved Examples :-

Example 1: Reduce the matrix to Echelon and find its rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ -1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ -1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2\,R_1$

$R_3 \Rightarrow R_3 - 3\,R_1$

$R_4 \Rightarrow R_4 - 6\,R_1$

$$A=\begin{bmatrix}1 & -1 & 2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix}$$

$R_3 \Rightarrow R_3 - \dfrac{4}{5}\ R_2$

$R_4 \Rightarrow R_4 - \dfrac{9}{5}\ R_2$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 0 & 33/5 & 22/5\\ 0 & 0 & 33/5 & 22/5\end{bmatrix}$$

$R_4 \Rightarrow R_4 - \ R_3$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 0 & 33/5 & 22/5\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\therefore$ Rank of A = $\mathrm{e}(\mathrm{A})$

$= No.\ of$ non-zero rows

$=\ 3$

Check Your Progress:

1) Find the rank of the following matrices by reducing to Echelon form.

i) $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 7\\ 3 & 6 & 10\end{bmatrix}$ $\quad Ans : 2$

ii) $A=\begin{bmatrix}1 & 2 & -1 & 3\\ 3 & 4 & 0 & -1\\ -1 & 0 & -2 & 7\\ 1 & 2 & 3 & -1\end{bmatrix}$

iii) $A=\begin{bmatrix}3 & 4 & 1 & 1\\ 2 & 4 & 3 & 6\\ -1 & -2 & 6 & 4\\ 1 & -1 & 2 & -3\end{bmatrix}$ Ans : 4

2.4 LINEAR ALGEBRIC EQUATIONS

i) Consider a set of equations :

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

The equation can be written in the matrix form as :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A X D

i.e. $\mathrm{AX}=\mathrm{D}$

Now we join matrices A and D

$$[A:D]=\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

It is called as Augment matrix

We reduce (A.D.) to Echelon form and thereby find the ranks of A and (A:D)

1) If $\rho(A)\neq\rho(AD)$ then the system is inconsistent i.e. it has no solution.

2) If $\rho(AD)=\rho(A)$ then the system is consistent and if

(i) $\rho(AD)=\rho(A)=$Number of unknowns then the system is consistent and has unique solution.

(ii) $\rho(AD)=\rho(A)<$ Number of unknowns and has infinitely many solutions.

Non- Homogeneous equation:-

System of simultaneous equation in the matrix form is AX=D…..(I)

Pre-multiplying both sides of I by A^{-1} we set

$\therefore A^{-1}AX = A^{-1}D$
$\therefore IX = A^{-1}B$
$\therefore X = A^{-1}B$
which is required solution of the given non-homogeneous equation.

Homogeneous linear equation:-

Consider the system of simultaneous equations in the matrix form.
$AX = D$
If all elements of D are zero

i.e

then the system of equation is known as homogeneous system of equations.

In this case coefficient matrix A and the augmented matrix [A,O] are the same. So The rank is same. It follow that the system has solution

$x_1, x_2, x_3 ....x_4 = 0,$ which is called a trivial solution.

Example 2: Solve the following system of equations

$2x_1 - 3x_2 + x_3 = 0$
$x_1 + 2x_2 - 3x_3 = 0$
$4x_1 - x_2 - 2x_3 = 0$
Solution: The system is written as

$$AX = 0$$
$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Hence the coefficient and augmented matrix are the same
We consider

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$
$R_1 \Rightarrow R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 4 & -1 & -2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 \text{-} 2\, R_1 \,\&\, R_3 \Rightarrow R_3 - 4R_1$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 7 \\ 0 & -9 & -10 \end{bmatrix}$$

$R_2 \Rightarrow R_2 \times 1/7$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -9 & -10 \end{bmatrix}$$

$R_3 \Rightarrow R_3 + 9R_2 \,\&\, R_1 \Rightarrow R_1 - 2R_2$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -19 \end{bmatrix}$$

$R_3 \Rightarrow R_3 \times -1/7$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \Rightarrow R_2 + R_3 \,\&\, R_1 \Rightarrow R_1 + R_3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence Rank of A is 3

$\therefore \ell(A) = 3,$

The coefficient matrix is non-singular

Therefore there exist a trivial solution

$x_1 = x_2 = x_3 = 0$

Example 3: Solve the following system of equations

$$x_1 + 3x_2 - 2x_3 = 0$$
$$2x_1 - x_2 + 4x_3 = 0$$
$$x_1 - 11x_2 + 14x_3 = 0$$

Solution: The given equations can be written as

$$AX = 0$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & 11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the coefficient & augmented matrix are the same

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$R_2 \Rightarrow R_2 \text{-} 2\,R_1 \,\&\, R_3 \Rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Here rank of A is 2 i.e

$$\ell(A) = 2$$

So the system has infinite non-trivial solutions.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 - 2x_3 = 0$

$-7x_2 - 8x_3 = 0$

$7x_2 = 8x_3$

$x_2 = \frac{8}{7} x_3$

Let $x_3 - 8x_3 = \lambda$

$\therefore x_2 = \frac{8}{7}\lambda$

$\therefore x_1 + 3\left(\frac{8}{7}\lambda\right) - 2\lambda = 0$

$\therefore x_1 + \frac{24}{7}\lambda - 2\lambda = 0$

$\therefore x_1 = 2\lambda - \frac{24}{7}\lambda$

$\therefore x_1 = -\frac{10}{7}\lambda$

Hence $x_1 = -\frac{10}{7}\lambda \quad x_2 = \frac{8}{7}\lambda$ and $x_3 = \lambda$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{10}{7}\lambda \\ \frac{8}{7}\lambda \\ \lambda \end{bmatrix}$$

Hence infinite solution as deferred upon value of λ

Example 4: Discuss the consistency of

$$\begin{aligned} 2x+3y-4z &= -2 \\ x-y+3z &= 4 \\ 3x+2y-z &= -5 \end{aligned}$$

Solution: In the matrix form

$$\begin{bmatrix} 2 & 3 & -4 \\ 1 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix}$$

Consider an Agumental matrix

$$[A:D] = \begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 1 & -1 & 3 & : & 4 \\ 3 & 2 & -1 & : & -5 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{1}{2}\ R_1$$

$$R_3 \to R_3 - \frac{3}{2}\ R_1$$

$$[A:D] = \begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 0 & -5/2 & 5 & : & 5 \\ 0 & -5/2 & 5 & : & -2 \end{bmatrix}$$

$$R_2 \to R_3 - R_2$$

$$[A:D] = \begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 0 & -5/2 & 5 & : & 5 \\ 0 & 0 & 5 & : & -7 \end{bmatrix}$$

$$\therefore\ \rho(AD)=3$$

$$\rho(A)=2$$

$$\therefore\ \rho(AD)\neq\rho(A)$$

$\therefore$ The system is inconsistent and it has no solution.

Example 5: Discuss the consistency of

$3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

Solution: In the matrix form,

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$A \qquad X = D$$

Now we join matrices A and D

Consider

$$[A:D] = \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

We reduce to Echelon form

$$R_1 \to R_3$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

$$R_2 \to R_2 - 2\ R_1$$

$$R_3 \to R_3 - 3\ R_1$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{5}{7}\ R_2$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8/7 & : & -8/7 \end{bmatrix} ....(1)$$

This is in Echelon form

$$\therefore \rho(AD) = 3$$

$$\rho(A) = 3$$

$\therefore\ \rho\ (\text{AD}) = \rho\ (\text{A}) =$ Number of unknowns

$\therefore$ system is consist and has unique solution.

Step (2) : To find the solution we proceed as follows. At the end of the row transformation the value of z is calculated then values of y and the value of x in the last.

The matrix in e.g. (1) in Echelon form can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 8/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -8/7 \end{bmatrix}$$

$\therefore$ Expanding by R_3

$$\frac{8}{7} Z = -8/7$$

$\therefore$ z = -1

$\therefore$ expanding by R_2

$$-7y - 3z = -11$$

$$-7y - 3(-1) = -11$$

$$-7y + 3 = -11$$

$$\not{-}7y = \not{-}14$$

$$y = 2$$

exp*anding* by R_1

$$x + 2y + z = 4$$

$$x + \not{4} - 1 = \not{4}$$

$\therefore$ x = 1

$\therefore$ x = 1, y = 2, z = -1

Example 6: Examine for consistency and solve

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$7x+2y+10z=5$

Solution:

Step (1) : In the matrix form

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$A \qquad X = D$$

Consider

$$[A:D] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$R_1 \to \frac{1}{5} R_1$$

$$[A:D] = \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1$$

$$R_3 \to R_3 - 7R_1$$

$$[A:D] = \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & -11/5 & 1/5 & : & -3/5 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{1}{11} R_2$$

$$[A:D] = \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\therefore \quad \rho\,(AD) = 2$

$\rho(A) = 2$

$\therefore \quad \rho(AD) = \rho(A) = 2 < 3 = Number$ of unknowns

The system is consistent and has infinitely many solutions.

Step (2) :- To find the solution we proceed as follows:

Let

$z = k ....[k = \text{ parameter}]$

$\therefore$ By expanding R_2

$121/5y - 11/5z = 33/5$

$\therefore$ 11y-z $= 3$

$\therefore$ $y = \frac{z+3}{11}$

$\therefore$ put $z = k$

$\therefore$ $y = \frac{k+3}{11}$

By exapanding R_1

$x + 3/5\, y + 7/5\, z = 4/5$

$\therefore$ $x = \frac{7}{11} - \frac{16k}{11}$

Check Your Progress:

Solve the system of equations:

i) $2x_1 + x_2 + 2x_3 + x_4 = 6$

$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

$2x_1 + 2x_2 - x_3 + x_4 = 10$

Ans : consistent

ii) $x_1 = 2,\ x_2 = 1,\ x_3 = -1,\ x_4 = 3$

$2x_1 + x_2 + x_3 + x_4 = 2$

$3x_1 - x_2 + x_3 - x_4 = 2$

$x_1 + 2x_2 - x_3 + x_4 = 1$

$6x_1 + 2x_2 + x_3 + x_4 = 5$

Ans : Infinitely many solutions,

iii) $x_1 = k,\ x_2 = 3 - 4k,\ x_3 = 2 - \frac{5}{2}k,\ x_4 = \frac{9}{2}k - 3$

3 $x_1 + x_2 + x_3 = 4$

$2x_1 + 5x_2 - 2x_3 = 3$

$x_1 + 7x_2 - 7x_3 = 5$

Ans : Inconsistent

iv) $x_1 - x_2 - x_3 = 0$

$x_1 + 2x_2 - x_3 = 0$

$2x_1 + x_2 + 3x_3 = 0$

Ans: Trivial Solution.

v) $x_1 + 2x_2 + 3x_3 = 0$

$2x_1 + 4x_2 + 7x_3 = 0$

$3x_1 + 6x_2 + 10x_3 = 0$

Ans : Definitely many solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda \\ \frac{-1}{2}\lambda \\ 0 \end{bmatrix}$$

2.5 LET US SUM UP

In this chapter we have learn

- ❖ Using row echelon from finding Rank of matrix.
- ❖ Representing linear equation m x n in to argumented matrix.
- ❖ Consistency of matrix.
- ❖ Solution of Homogeneous equations.
- ❖ Solution of non homogeneous equations.

2.6 UNIT END EXERCISE

1) Reduce the following matrix in Echolon form & find its Rank.

i) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ *Ans : Rank = 2*

ii) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ *Ans : Rank = 3*

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ *Ans : Rank = 2*

iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ *Ans : Rank = 2*

2) Solve the following system of equations.

i) $x_1 + x_2 + x_3 = 3,\ x + 2x_2 + 3x_3 = 4,\ x_1 + 4x_2 + 9x_3 = 6$

Ans:- $x = 2, y = 1, z = 0.$

ii) $2x_1 - x_2 - x_3 = 0,\ x_1 - x_3 = 0,\ 2x_1 + x_2 - 3x_3 = 0$

Ans:- $x_1 = x_2 = x_3 = \lambda ..... \therefore \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

iii) $5x_1 - 3x_2 - 7x_3 + x_4 = 10$

$-x_1 + 2x_2 + 6x_3 - 3x_4 = -3$

$x_1 + x_2 + 4x_3 - 5x_4 = 0$

iii) $2x_1 + 3x_2 - 2x_3 = 0$

$3x_1 - x_2 + 3x_3 = 0$

$7x_1 + 5x_2 - x_3 = 0.$

iv) $x_1 - 4x_2 - x_3 = 3$

$3x_1 + x_2 - 2x_3 = 7$

$2x_1 - 3x_2 + x_3 = 10.$

v) $x_1 - 4x_2 + 7x_3 = 8$

$3x_1 + 8x_2 - 2x_3 = 6$

$7x_1 - 8x_2 + 26x_3 = 31$

*****

3

LINEAR DEPENDANCE AND INDEPENDANCE OF VECTORS

UNIT STRUCTURE

3.1 Objectives

After going through this chapter you will able to

- ❖ Find linearly independent & linearly dependent vector.
- ❖ Inner product of two vector
- ❖ Find characteristic equation of matrix
- ❖ Find the of characteristic equation i.e
- ❖ Find the corresponding .Eigen vector to Eigen value.

3.2 Introduction

In this chapter we are going to discuss linearly dependent & independent also. Inner two vector using the characteristic equation of matrix. We are going to evaluate .Eigen value & Eigen.vector of matrix A.

Vector :- An set of n elements written as $x=\left[x_1,x_2,x_3,x_4,.............x_n\right]$ is called a vector of n-dimensions.

Note : Any two or column matrix is called as a vector and numbers are called as scalars.

3.3 Definitions

Linearly Independent Vector

Let

$Let\ x_1,x_2,.............x_n$ be n vectors of some order

Let $c_1x_1 + c_2x_2 + ... + c_nx_n = 0$

Where $c_1,\ c_2,$ ...... are scalars.

If (i) $c_1 = c_2 = ... = c_n = 0$ then

$x_1,\ x_2,\ ....x_n$ are linearly independent

and (ii) if not all c_i are zero then $x_1, x_2, ... x_n$ are linearly dependent

If $x_1, x_2, ...x_n$ are linearly dependent then a relation exists between them which can be found out

Solved examples:-

Example 1: Examine for linear dependence

$x_1 = (1\ \ 2\ \ 4)^T,\ \ x_2 = (3\ \ 7\ \ 10)^T$

Solution: We have,

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},\quad x_2 = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$$

$$Let\ \ c_1x_1 + c_2x = 0$$

$$i.e.\quad c_1\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e.\quad \begin{bmatrix} c_1 & + & 3c_1 \\ 2c_1 & + & 7c_2 \\ 4c_1 & + & 10c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\quad c_1 + 3c_2 = 0$$

$2c_1 + 7c_2 = 0$

$4c_1 + 10c_2 = 0$

Consider first two equations in matrix form.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\quad X\quad = 0$$

$$\therefore\quad |A| = 7 - 6$$

$|A| = 1$

$\therefore\quad |A| \neq 0$

$\therefore$ system has zero solution.

i.e. $\quad c_1 = c_2 = 0$

$\therefore\quad x_1,\ x_2$ are linearly independent

Example 2: Examine for linear dependence.

$x_1 = (1 \;\; 2 \;\; 3)^T$, $x_2 = (3 \;\; \text{-2} \;\; 1)^T$, $x_3 = (1 \;\; \text{-6} \;\; \text{-5})^T$

Solution:

Step (1) We have

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \;,\; \mathrm{x}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} , \; \mathrm{x}_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$$

$$Let \quad \mathrm{c}_1 x_1 + \mathrm{c}_{21} x_2 + \mathrm{c}_3 x_3 = 0$$

$$\therefore \quad c_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mathrm{c}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + \mathrm{c}_3 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1 & +3c_2 & +c_3 \\ 2c_1 & -2c_2 & -6c_3 \\ 3c_1 & +c_2 & -5c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad c_1 + 3c_2 + c_3 = 0$$

$$2c_1 - 2c_2 - 6c_3 = 0$$

$$3c_1 + c_2 - 5c_3 = 0$$

Step (ii) In matrix form,

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \\ \mathrm{c}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{A} \qquad \mathrm{X} \quad = \quad 0$$

Consider

$$[\mathrm{A}{:}0] = \begin{bmatrix} 1 & 3 & 1 & : & 0 \\ 2 & -3 & -6 & : & 0 \\ 3 & 1 & -5 & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2\,R_1$

$R_3 \to R_3 - 3\,R_1$

$$[\mathrm{A}{:}0] = \begin{bmatrix} 1 & 3 & 1 & : & 0 \\ 0 & -8 & -8 & : & 0 \\ 0 & -8 & -8 & : & 0 \end{bmatrix}$$

$R_3 \to R_3 - \; R_2$

$R_2 \to -\dfrac{1}{8}\; R_2$

$$[\mathrm{A}{:}0] = \begin{bmatrix} 1 & 3 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$e\;(\mathrm{A}\;0) = 2$

$e\ (A) = 2$

$\therefore \quad e\ (A{:}0) = e\ (A) = 2 <$ Number of unknowns

$\therefore$ system has non-zero solution

i.e. $c_1,\ c_2,\ c_3$ are non zero

$\therefore\ x_1, x_2, x_3$ are linearly dependent

Step (iii):

To find relation between

$x_1,\ x_2,\ x_3$

Let

$c_3 = k$

By expanding R_2

$c_2 + c_3 = 0$

$\therefore \quad c_2 = -c_3$

$c_2 = -k$

By expanding R_1

$c_1 + 3c_2 + c_3 = 0$

$c_1 - 3k + k = 0$

$c_1 = 2k$

$\therefore\ c_1x_1 + c_2x_2 + c_3x_3 = 0$

$\therefore\ 2kx_1 - kx_2 + kx_3 = 0$

$\therefore\ 2x_1 - x_2 + x_3 = 0$ is a relation.

Check your progress:

1) Show that the vectors $x_1 = (1\ 1\ \ 1\ ),\ x_2\,(1,\ 2,\ 3), x_3\,(\ 2,\ 3,\ 8)$ are linearly independent

2) Are the following vectors linearly dependent? If so find the relation

i) $x_1 = (1\ 2\ \ 4\ ),\ x_2 = (2,\ -1,\ 3), x_3 = (\ 0,\ 1,\ 2), x_4 = (\ -3,\ 7,\ 2)$

Ans : Dependent $\quad 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

(ii) $x_1 = (2\ \ -1\ \ 3\ \ 2),\ x_2 = (1\ \ 3\ \ 4\ \ 2),\ x_3 = (3\ \ -5\ \ 2\ \ 2)$

Ans :- Dependent, $2x_1 - x_2 - x_3 = 0$

(iii) $x_1 = (1\ \ 1\ \ 1\ \ 3),\ x_2 = (1\ \ 2\ \ 3\ \ 4),\ x_3 = (2\ \ 3\ \ 4\ \ 9)$

Ans : Independent.

3.4 THE INNER PRODUCT

If $X=(x_1,x_2.......x_n)$ and $Y=(y_1,y_2.......y_n)$
then<X,Y> denotes inner product
$<X,Y>=x_1y_1+x_2y_2+x_3y_3+........+x_ny_n$ is in inner product of X and Y.
Let V be a vector space and X,Y V then<X,Y> it said to be an inner product if it satisfies following properties.

i) <X,Y> =0

ii) <X,Y> = <Y,X>

iii) <X,Y+Z> = <X,Y> + <X,Z>

iv) <X, α Y> = α <X,Y> where α is scalar.

v) <X,Y> = 0 if and only if X=0.

Example 3: Show that $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$
Satisfies all properties of inner product

Solution: $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$

i) $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$

$$=(x_1)^2+2(x_2)^2+4(x_3)^2\geq 0$$

$$<X,Y>\geq 0$$

$$<X,Y>=0(x_1)^2+2(x_2)^2+4(x_3)^2=0$$

$$x_1=0, x_2=0, or x_3=0$$

$$\therefore <X,X>0x=0$$

ii) $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$

$$=y_1x_1+2y_2x_2+4y_3x_3$$

$$=<Y,X>$$

iii) $<X,Y+Z>=x_1(y_1+z_1)+2x_2(y_2+z_2)+4x_3(y_3+z_3)$

$$=x_1y_1+x_1z_1+2x_2y_2+2x_2z_2+4x_3y_3+4x_3z_3$$

$$=x_1y_1+2x_2y_2+4x_3y_3+x_1z_1+2x_2z_2+4x_3z_3$$

$$=<X,Y>+<X,Z>$$

iv) $<X,\alpha Y> \ = x_1(\alpha y_1)+2x_2(\alpha y_2)+4x_3(\alpha y_3)$

$$=\alpha x_1y_1+\alpha 2x_2y_2+\alpha 4x_3y_3$$

$$=\alpha(x_1y_1+2x_2y_2+4x_3y_3)$$

$$=\alpha<x,y>$$

Here all properties are satisfied
$\therefore <X,Y>$ is an inner product.

Check Your Progress:

Prove all the properties of an inner product for the following:-

i. $<X,Y>=16x_1y_1-25x_2y_2$

ii. $<X,Y>=8x_1y_1+x_2y_2-x_3y_3$

iii. $< X, Y > = 3x_1y_1 - x_2y_2 - 4x_3y_3$

iv. $< f, g > = \int_a^b f(t).g(t).dt$

3.5 Eigen Values And Eigen Vectors

Definition:-

Let A be a given square matrix.
Then there exists a scalar λ and non-zero vector X such that

$AX = \lambda X$..........(1)

Our aim is to find and x for given matrix A using equation (1)

λ is called as eigen value, latent roots of a matrix value, characteristic value or root of a matrix A and x is called as eigen vector or characteristic vector etc.

X is a column matrix

Method of finding λ and x :-
We have,

$$AX = \lambda X$$

$$\therefore \quad AX\text{-}\lambda\, IX = 0..... \left[x = IX,\ I = \text{unit matrix}\right]$$

$$\therefore \left(A\text{-}\lambda\, I\right)X = 0.............(2)$$

Equation 2 is a set of homogenous equation and for non-zero x, we have

$$\left|A - \lambda\, I\right| = 0..........(3)$$

This equation is called the characteristic equation of
First we solve equation (3) to find eigen values or roots. Then we solve equation (2) to find Eigen vectors.
Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

equation (2) i.e. $\left(A\text{-}\lambda\, I\right)x = 0$ becomes

$$\left\{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e.\begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\rightarrow(2)$$

and equation (3) i.e. $|\text{A-}\lambda\text{ x}|=0$ is

$$\begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix}=0\rightarrow(3)$$

Note :

1) Equation (2) is called as matrix equation of A in λ
2) Equation (3) is called as characteristic equation of A in λ
3) Usually given matrix A is of order 3X3 . Therefore it will have 3 eigen values and for every eigen value there will be corresponding eigen vector which is a column matrix of order 3X1. There are 3 such column matrices.
4) Eigen vectors are linearly independent.
5) Method of finding eigen values is same for any given matrix A.

Method of finding eigen vectors is slightly different and we study 3 types of such problems.

Type (I) : When all eigen values are distinct and matrix A may be symmetric or non- symmetric.
Type (II) : When eigen values are repeated and A is non-symmetric
Type (III) : When eigen values are repeated and A is symmetric.

Solved examples :-

Type (I) : All roots are non- repeated.

Example 4: Find eigen values and given vectors for

$$A=\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution: *Step* (1) : Charactristic equation of A in λ i s

$$|\text{A-}\lambda\text{ I}|=0$$

$$i.e.\begin{vmatrix} 2\text{-}\lambda & \text{-}2 & 3 \\ 1 & 1\text{-}\lambda & 1 \\ 1 & 3 & \text{-}1\text{-}\lambda \end{vmatrix}=0$$

$$\therefore\ \lambda^3-(\textit{sum of}\text{ diagonal elements of A})\ \lambda^2+$$

$$(\textit{sum of}\text{ minors of diagonal elements of A})\ \lambda\ \text{-}|\text{A}|=0$$

$$\therefore|A|=2\ (-1-3)+2\ (-1-1)+3\ (3-1)$$

$= -8-4+6$

$|A| = -6$

Characteristic equation is given by

$$\therefore \lambda^3 - 2\lambda^2 + (-4-5+4)\lambda - (-6) = 0$$

$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

since sum of coefficient $= 0$

$\therefore (\lambda - 1)$ is a factor.

Synthetic division:

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$\therefore (\lambda - 1)(\lambda^2 - \lambda - 6) = 0$

$\therefore (\lambda - 1)(\lambda - 3) \cdot (\lambda + 2) = 0$

$\therefore \lambda = 1, -2, 3$

$\therefore$ The roots are non- repeated

Step (ii) :- Now we find eigen vectors

Matrix equations is given by

$$(A - \lambda I)X = 0$$

$$i.e. \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :– When $\lambda = 1$, matrix eq^n becomes

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving first two rows by Cramer's rule.

We have,

$$x_1 - 2x_2 + 3x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$

$$\therefore \frac{x_1}{-2} = \frac{-x_2}{-2} = \frac{x_3}{-2}$$

$$\therefore \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore\ x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) :- When $\lambda_2 = -2$

Matrix equation is given by

$$\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ \frac{x_1}{-11} = \frac{x_2}{1} = \frac{x_3}{14}$$

$$\therefore\ \frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14}$$

$$\therefore\ x_2 = \begin{bmatrix} -11 \\ -1 \\ 14 \end{bmatrix}$$

Case (iii) : When $\lambda_3 = +3$ matrix equation is given by

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\therefore\ \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{4} \quad \therefore\ x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Type (II) :- Repeated eigen values and A is non- symmetric.

Example 5: Find eigen values and eigen vectors for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution:

Step (1) :- Characteristic equation of A in λ is

$$[A - \lambda\ I] = 0$$

$$i.e.\ \ \lambda^3 - 9\lambda^2 + (6+5+4)\lambda - 7 = 0$$

$$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

since sum of co-efficients $= 0$

$\therefore\ (\lambda - 1)$ is a factor

synthetic division

1	1	-9	15	-7
		1	-8	7
	1	-8	7	0

$\therefore \lambda^2 - 8\lambda + 7$

$= (\lambda - 7)(\lambda - 1)$

$\therefore \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

$\therefore (\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$

$\lambda = 7, 1, 1$

Here two roots are repeated. First we find eigen vectors for non-repeated root.

Step II :- Matrix equation of A in λ is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- For $\lambda = 7$

Matrix equation is

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Case (ii) :- Let $\lambda = 1$

Matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By cramers's rule we get

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$

$$i.e. \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But by definition we want non-zero x_2

So we proceed as follows

$Expanding$ by R_1

$x_1 + x_2 + x_3 = 0$

Assume any element to be zero say x_1 and give any conventional value say 1 to x_2 and find x_3

Let

$x_1 = 0, \ \ x_2 = 1$

$\therefore \ x_3 = -1$

$$\therefore \ x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (iii) :- Let x=1

Again consider

$x_1 + x_2 + x_3 = 0$

$Let \quad x_2 = 0, \ \ x_1 = 1$

$\therefore \ x_3 = -1$

$$\therefore \ x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Type (iii) :- A is symmetric and eigen values are repeated

Example 6: Find eigen values and eigen vectors for .

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution:

Step :- Characteristic equations of A in λ is

$$[A - \lambda I] = 0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$[A] = 32$

$i.e.\ \lambda^3 - 12\lambda^2 + (8+14+14)\ \lambda \text{ -}32 = 0$

$\therefore \ \lambda^3 - 12\lambda^2 + 36\ \lambda \text{ -}32 = 0$

$(\lambda \text{ -}2)$ is a factor

Synthetic division :-

2	1	−12	36	−32
		2	−20	32
	1	−10	16	0

$$\lambda^2 - 10\lambda + 6$$
$$= (\lambda - 8)(\lambda - 2)$$
$$\therefore\ \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$
$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

$\therefore\ \lambda = 8,\ 2,\ 2$

Step (ii) :- Matrix equation is

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- For $\lambda = 8$

Matrix equation is given by

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ \frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$ ....*By* cramer's rule

$$\frac{x_1}{2} = \frac{-x_2}{-1} = \frac{x_3}{1}$$

$$\therefore\ x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii) :- Let $\lambda = 2$

Matrix equation is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding R_1

$4x_1 - 2x_2 + 2x_3 = 0$

Let $x_1 = 0,\ x_2 = 1$

$\therefore\ x_3 = 1$

$$\therefore\ x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) :- Let

$\lambda = 2$

$\therefore$ A is symetric

$\therefore\ x_1,\ x_2,\ x_3$ are orthogonal

Let, $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$\because$ x_1, x_3 are orthogonal

$\therefore$ $x_1^1, x_3 = 0$

$\therefore$ $2l - m + n = 0$..........(1)

x_2, x_3 are orthogonal

$\therefore$ $x_2^1, x_3 = 0$

$\therefore$ $ol + m + n = 0$..........(2)

solving (1) and (2) by cramer's rule

$$\frac{l}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\therefore \frac{l}{+1} = \frac{m}{1} = \frac{n}{-1}$$

$$\therefore x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Check your progress:

1) Find eigen values and eigen vectors for

i) $A = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Ans :- Eigen values are 0,1,2

$$\therefore x = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Ans *Eigen values* are 2,3 and 6

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(iii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{bmatrix}$

Ans : *Eigen values* are 5, -3,-3

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \mathrm{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \mathrm{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

3.6 SUMMARY

In this chapter we have learn

- ❖ Linearly dependent & independent vector.
- ❖ Inner product of two vector i.e same as dot product 7 its properties.
- ❖ Characteristics equation & its root by using

$|A - \lambda I| = 0$

- ❖ Eigen vector which is corresponding to Eigen value which we get from $|A - \lambda I| = 0$

3.7 UNIT END EXERCISE

1) Is the system of vector $x_1 = (2,2,1)^T, x_2 = (1,3,1)^T$ linear by dependent?

2) Show that the vectors (1,2,3) (2,20) form a linearly independent set.

3) Show that the following vector are linearly dependent & find the relation between them

$x_1 = (1,-1,1), x_2 = (2,1,1), x_3 = (3,0,2)$

4) Prove the properties of an inner product.

i. $< X, Y >= 3x_1y_1 + 4x_2y_2$.

ii. $< X, Y >= 9x_1y_1 - 3x_2y_2 + 4x_3y_3$

5) Find Eigen value and Eigen vector for the following matrix.

i) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

iv) $A = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$

v) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

vi) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

vii) $A = \begin{bmatrix} Cos0 & -Sin\theta \\ Sin\theta & Cos0 \end{bmatrix}$

viii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

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4

CAYLEY – HAMILTON THEORY

UNIT STRUCTURE

4.1 OBJECTIVE

After going through this chapter you will able to

- ❖ Find by using Cayley Hamilton Theorem.
- ❖ Application of Cayley- Hamilton Theorem.
- ❖ Find diagonal matrix on similar matrix.
- ❖ Characteristic Polynomial & Minimal Polynomial of matrix A.
- ❖ Derogatory & non-derogatory matrix.
- ❖ Complex matrix like Hermitian, Skew-Hermitian unitary matrix.

4.2 INTRODUCTION

In previous chapter we learn about Eigen values & Eigen Vector. How here we are going to discuss Cayley Hamilton Theory & its application also we had study only Real matrix. We introduce here complex matrix with type of complex matrix also minimal polynomial.

4.3 CAYLEY – HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation.
If the characteristic Equation for the n^{th} order square matrix A is

$|A- \lambda I| = (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} .... + a_n\right]$ then

$(-1)^n \left(A^n + a_1A^{n-1} + a_2A^{n-2} .... + a_nI\right) = 0$

Example 1:

Show that the given matrix A satisfies its characteristic equation.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

The characteristic equation of the matrix A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore (2-\lambda)\left[(1-\lambda)(2-\lambda)-0\right]-1(0)+1\left(0-(1-\lambda)\right)=0$$

$$\therefore (2-\lambda)\left[2-3\lambda+\lambda^2\right]+1(-1+\lambda)=0$$

$$\therefore 4-6\lambda+2\lambda^2-2\lambda+3\lambda^2-\lambda^3-1+\lambda=0$$

$$\therefore -\lambda^3+5\lambda^2-7\lambda+3=0$$

$$\therefore \lambda^3-5\lambda^2+7\lambda-3=0$$

By Cayley Hamilton theorem,

$$A^3-5A^2+7A-3I=0 \text{……………………….(1)}$$

Now, we have

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$\therefore A^3-5A^2+7A-3I =$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} -5 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} +7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} -5 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} +7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 20 & 20 \\ 0 & 8 & 0 \\ 20 & 20 & 28 \end{bmatrix} - \begin{bmatrix} 28 & 20 & 20 \\ 0 & 8 & 0 \\ 20 & 20 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$$\therefore A^3 - 5A^2 + 7A - 3I = 0$$

Thus the matrix A satisfies its characteristic equation.

Example 2 :

Calculate A^7 by using Cayley Hamilton theorem.

Where A = $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

Solution :

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\ (2-\lambda) - 6 = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 - 6 = 0$$

$$\therefore\ \lambda^2 - 5\lambda = 0$$

By Cayley Hamilton theorem,

$$A^2 - 5A = 0$$

i.e. $A^2 = 5A$

Now to calculate

$$A^7 = A^5.A^2 = A^5.5A = 5A^6$$
$$= 5A^4.A^2 = 25A^5$$
$$= 25A^3.A^2 = 125A^4$$
$$= 125A^2.A^2 = 125(5A).(5A)$$
$$= 3125A^2 = 3125(5A)$$
$$= 15625A$$

$$A^7 = 15625\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 46875 & 93750 \\ 15625 & 31250 \end{bmatrix}$$

$\therefore$ The value of $A^7 = \begin{bmatrix} 46875 & 93750 \\ 15625 & 31250 \end{bmatrix}$

Example 3:

By using Cayley Hamilton theorem find A^{-1}

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

The characteristics equation of A is

$$|A - \lambda I = 0|$$

$$\begin{bmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left[1-2\lambda+\lambda^2-4\right]+1[\lambda-1-2]+1[-2+\lambda-1]=0$$

$$\lambda^2-2\lambda-3+3\lambda+2\lambda^2-\lambda^3+\lambda-3-3+\lambda=0$$

$$-\lambda^3+3\lambda^2+3\lambda-9=0$$

$$\lambda^3-3\lambda^2-3\lambda+9=0$$

By Cayley Hamilton theorem

$$A^3-3A^2-3A+9I=0$$

Multiply by A^{-1}

$$\therefore\ A^3A^{-1} - 3A^2A^{-1} - 3AA^{-1} + 9IA^{-1} = 0A^{-1}$$

$$\therefore\ A^2 - 3A - 3I + 9A^{-1} = 0$$

$$A^{-1} = \frac{1}{9}\left[3A + 3I - A^2\right] \qquad \text{.......(1)}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$3A + 3I - A = 3\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$3A + 3I - A^2 = \begin{bmatrix} 3 & -3 & 3 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 3 \\ -3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left[3A + 3I - A^2\right]$$

$$= \frac{1}{9}\begin{bmatrix} 3 & -3 & 3 \\ -3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Check your progress:

1) Find the characteristic polynomial of the matrix.

$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Verify Cayley-Hamilton theorem for this matrix.

Hence find A^{-1}

Ans: $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$

2) Use Cayley-Hamilton theorem to find inverse of the matrix.

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$ Ans: $\frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$

3) Use Cayley-Hamilton theorem to find the inverse of

$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$

4) Show that the following matrices satisfy their characteristics equation

$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

5) Using the characteristics equation show that inverse of the matrix

i) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

4.4 SIMILARITY OF MATRIX

Two matrix A and B of order nxn over F are said to be similar if there exist a non-singular matrix P (invertible matrix) of order nxn such that $B = P^{-1}AP$

This transformation of matrix A by a non-singular matrix P to B is called a similarity transformation.

Diagonal matrix: If a square matrix A of order n has linearly independent eigen vectors then matrix P can be formed such that $P^{-1}AP$ is diagonal matrix i.e.
$D = P^{-1}AP$

Example 4:

Two similar matrices A and B have the same eign values.

Solutions:

Since A and B are similar, there exists a non-singular matrix P such that
$B = P^{-1}AP$

Consider $|B-\lambda I| = |P^{-1}AP - \lambda I|$

$$|B-\lambda I| = |P^{-1}AP - \lambda P^{-1}IP|$$
$$= |P^{-1}(A-\lambda I)P|$$
$$= |P^{-1}||A-\lambda I||P|$$
$$= |A-\lambda I||P^{-1}||P|$$
$$= |A-\lambda I| \qquad \because \ |P^{-1}|.|P| = 1$$
$$\therefore |B-\lambda I| = |A-\lambda I|$$

Hence the characteristics equation of A and B are the same
$\therefore$ A and B have same eigen values.

Example 5:

Show that $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ have same characteristics equations but A and B not similar matrices.

Solutions:

Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

Characteristics equation of A is $|A-\lambda I| = 0$

i.e. $\therefore \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1 = 0$ s equation

$\therefore$ Characteristics equation of B is

$|B-\lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1 = 0$

$\therefore$ Characteristics equation of A = Characteristics equation of B

Now we will show that A and B are not similar
Suppose $A \sim B$

$\therefore$ There exist non-singular matrix C such that, $B = 0\ C^{-1}AC$

Let $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2, \ \therefore$ C is non-singular as $= [C] \neq 0$

$\therefore C^{-1}$ exists

adj $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$C^{-1} = \frac{1}{[C]}$ adj $(C) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$C^{-1}AC = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq B$

Hence A and B are not similar matrices.

Example 6: Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, Find similarity to a diagonal matrix.

Find the diagonal matrix.

Ans : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

4.5 CHARACTERISTICS POLYNOMIAL

Solving the determinant $\left[A - \lambda I\right]$, a polynomial is obtained which is called as a characteristics polynomial.

For e.g. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristics polynomial is given by

$$|\propto||A - \lambda I| = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 1\right] + 1\left[-(2-\lambda)+1\right] + 1\left[1-(2-\lambda)\right]$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + 3\right] + 2\lambda - 2 - \lambda^3 + 6\lambda^{2-}3\lambda + 4$$

$$= \lambda^3 - 6\lambda^2 + 9\lambda - 4$$

4.6 MINIMAL POLYNOMIAL

Monic Polynomial: A Polynomial in λ, in which the coefficient of the highest power of λ is unity is called a monic polynomial.

For e.g. $\lambda^5 + 2\lambda^4 + 3\lambda^3 - 6\lambda + 5$ is a monic polynomial of degree polynomial.

If a polynomial f annihilates A then α f also f annihilates. A for $\alpha \in R$, therefore there exists a monic polynomial annihilating A.

If the characteristics roots of the characteristics equation are distinct then f (λ) = 0 is called minimal equation.

If matrix of order 3x3 are having characteristics root 2,3,3 then,

$$(\lambda - 2)(\lambda - 3) = 0$$

Or $(A-2)(A-3) = 0$ is the minimal equation.

Hence the degree of the equation is 2 and less than the order of the polynomial.

Derogatory Matrix: A nxn matrix is called derogatory if the degree of its minimal polynomial is less than n.

Non-Derogatory Matrix: A nxn matrix is called non-derogatory if the degree of minimal polynomial is equal to n.

Properties of Minimal Polynomial:

(1) There exists a unique minimal polynomial of the matrix A.

(2) The minimal polynomial of A divides the characteristics polynomial of A.

(3) If λ is the root of the minimal polynomial of A then λ is also characteristics of root of A.

(4) If the n characteristics of root of A are distinct then A is non derogatory.

Example 7:

Check whether the following matrix is derogatory or non derogatory also find its minimal polynomial.

i) $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution:

The characteristics polynomials of matrix A is

$$|A-\lambda I| = \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}$$

$$= \lambda^3 - (sum\ of\ diagonal\ element\ of\ A)\lambda^2 +$$

$$(sum\ of\ minor\ of\ diagonal\ element\ of\ A)\lambda - |A|$$

$$= \lambda^{3-}[2+1-1]\lambda^2 + \left[\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\right]\lambda - \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$\therefore \lambda^3 - 2\lambda^2 + [4-4-5]\lambda - (-6)$$

$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6$
$\therefore (\lambda+2)(\lambda-1)(\lambda-3)$

$\therefore$ The characteristics roots are -2, 1 and 3 which are distinct.

Therefore matrix A is non-derogatory.

ii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution:

The characteristics polynomials of matrix A is

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix}$$

$= \lambda^3 - (sum\ of\ diagonal\ element\ of\ A)\lambda^2 +$
$(sum\ of\ minor\ of\ diagonal\ element\ of\ A)\lambda - |A|$

$$= \lambda^{3-}[2+3+4]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}\right] - \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix}$$

$= \lambda^3 - 9\lambda^2 + [6+5+4]\lambda - 7$

$= \lambda^3 - 9\lambda^2 + 15\lambda - 7$

$= (\lambda-1)(\lambda-1)(\lambda-7)$

$\therefore$ The characteristics roots are 1, 1 and 7 which are not distinct.

Therefore matrix A is derogatory.

Example 8:

Show that the matrix A is derogatory also find its minimal polynomial.

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Solution:

The characteristics polynomials of matrix A is

$$|A-\lambda I| = \begin{bmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{bmatrix}$$

$$= \lambda^3 - \left(sum\ of\ diagonal\ element\ of\ A\right)\lambda^2 +$$

$$\left(sum\ of\ minor\ of\ diagonal\ of\ matrix\ A\right)\lambda - |A|$$

$$= \lambda^{3-}[1+4-3]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}\right]\lambda - \begin{vmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{vmatrix}$$

$$= \lambda^3 - 2\lambda^2 + [0-3+4]\lambda - 0$$

$$= \lambda^3 - 2\lambda^2 + \lambda$$

$$= \lambda(\lambda^2 - 2\lambda + 1)$$

$$= \lambda(\lambda-1)(\lambda-1)$$

$\therefore$ The characteristics roots are 0, 1 & 1 which are not distinct.

Therefore matrix A is derogatory matrix.

But we know that characteristic root of A is root of minimal polynomial.

$\therefore f(\lambda) = \lambda(\lambda-1) = \lambda^2 - \lambda.$

Now check whether $= f(\lambda)$ .annihilated matrix A.

$\therefore f(\lambda) = A^2 = A$

$$A^2 = A.A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$$A^2 - A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} - \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$A^2 - A = 0$

$\therefore \text{ f}(A) = 0$

$\therefore$ The minimal of polynomial of A is $\text{f}(\lambda) = \lambda^2 - \lambda$

& degree of polynomial is 2 which is less than 3
Hence matrix A is derogatory.

Example 9:

Find the minimal polynomial and show that it is derogatory matrix.

Where, $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution:

The characteristics polynomials of matrix A is

$$|A-\lambda I| = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right]-2\left[2-\lambda-1\right]+1(2-3+\lambda)$$

$$= (2-\lambda)\left[\lambda^2-5\lambda+6-2\right]-2\left[-\lambda+1\right]+\lambda-1$$

$$= -\lambda^3+5\lambda^2-4\lambda+2\lambda^2-10\lambda+8-3\lambda-3$$

$$= -\lambda^3+7\lambda^2-11\lambda+5$$

$$= (\lambda-1)(\lambda-1)(\lambda-5)$$

$\therefore$ The characteristics roots of matrix A are 1, 1 and 5.

$\because$ roots are.

$\therefore$ The matrix A is derogatory.

But we know that characteristics root of A is also a root of its minimal polynomial.

$$= \therefore\ f(\lambda) = (\lambda-1)(\lambda-5) = \lambda^2-6\lambda+5$$

Now check whether $f(\lambda)$ annihilated matrix A i.e.

$$f(A) = A^2-6A+5I = 0 ....(I)$$

$$A^2-6A+5I = \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} - 6\begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} - \begin{vmatrix} 12 & 12 & 6 \\ 6 & 18 & 6 \\ 6 & 12 & 12 \end{vmatrix} + \begin{vmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

$\therefore\ f(A) = 0$

$\therefore$ The minimal of polynomial of A is $f(\lambda) = \lambda^2 - 6\lambda + 5$

And degree of polynomial is 2 which is less than 3

$\therefore$ The matrix A is derogatory.

Check Your Progress:

(1) Show that the following matrices are derogatory and hence find the minimal polynomial.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ Ans: $\lambda^2 - 3\lambda + 2 = 0$

(ii) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$ Ans: $\lambda^2 - \lambda = 0$

(2) Check whether the following matrix is derogatory or non-derogatory also find the minimal polynomial.

(i) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ Ans: Non – derogatory

(ii) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Ans: Derogatory

4.7 COMPLEX MATRICES

Z =x+iy is called a complex number, where $i=\sqrt{-1}$ and $x, y \in R$ and $\bar{Z}=x-iy$ is called a conjugate of the complex number Z

Let A be a mxn matrix having complex numbers as its elements, then the matrix is called a complex matrix.

Conjugate of a Matrix:

The matrix of order mxn is obtained by replacing the elements by their corresponding conjugate elements, is called conjugate of a matrix. It is denoted by $\bar{A}$

For e.g. $A=\begin{vmatrix} 2-3i & 1-i & 3 \\ 2i+1 & 2 & 2i-3 \end{vmatrix}$

$$\bar{A}=\begin{vmatrix} 2+3i & 1+i & 3 \\ -2i+1 & 2 & -2i-3 \end{vmatrix}$$

Properties of conjugate of matrix:

(1) $\overline{(\bar{A})}=A$

(2) $\overline{A+B}=\bar{A}+\bar{B}$

(3) $\overline{(AB)}=\bar{A}.\bar{B}$

Conjugate Transpose:

Transpose of the conjugate matrix A is called conjugate transpose. It is denoted by A^{θ}.

For e.g. $A=\begin{vmatrix} 1+i & -i & 1 \\ 3 & i+2 & 3i-2 \end{vmatrix}$

$\bar{A}=\begin{vmatrix} 1-i & i & 1 \\ 3 & -i+2 & -3i-2 \end{vmatrix}$ then $A^{\theta}=\begin{bmatrix} 1-i & 3 \\ i & -i+2 \\ 1 & -3i-2 \end{bmatrix}$

Properties of Transpose of Conjugate of a matrix:

(1) $\left(A^{\theta}\right)^{\theta}=A$

(2) $(A+B)^{\theta}=A^{\theta}+B^{\theta}$

(3) $(AB)^{\theta}=B^{\theta}.A^{\theta}$

Hermitian matrix:

A square matrix A is called Hermitian matrix if A= A^{θ} i.e. A= $A=\left[a_{ij}\right]_{m\times n}$ is Hermitian if $a_{ij}=\bar{a}_{ji}$ $\forall$ i and j.

Example 10:

Show that the matrix $A=\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & -i \\ 3+i & i & 3 \end{bmatrix}$ is Hermitian

Solution:

Here $A=\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & -i \\ 3+i & i & 3 \end{bmatrix}$

$\bar{A}=\begin{bmatrix} 1 & 2+i & 3+i \\ 2-i & 3 & i \\ 3-i & -i & 3 \end{bmatrix}$

$A^{\theta}=\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & -i \\ 3+i & i & 3 \end{bmatrix}$

$\therefore A=A^{\theta}$

Hence by definition A is Hermitian matrix.

Skew Hermitian Matrix:

A Square matrix A such that $A^{\theta}=-A$ is called a Skew Hermitian Matrix.
i.e. if $A=\left[a_{ij}\right]_{m\times n}$ is Skew Hermitian if $a_{ij}=-\bar{a}_{ji}$ $\forall$ i and j.
Here a_{ij} = purely imaginary or re a_{ij} = 0.

Example 11:

Show that the matrix $A=\begin{bmatrix} 2i & 5+i & 6+i \\ -5+i & 0 & -i \\ -6+i & -i & 0 \end{bmatrix}$ is called a Skew Hermitian Matrix.

Solution:

Here $A=\begin{bmatrix} 2i & 5+i & 6+i \\ -5+i & 0 & -i \\ -6+i & -i & 0 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -2i & 5-i & 6-i \\ -5-i & 0 & i \\ -6-i & i & 0 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} -2i & -5-i & -6-i \\ 5-i & 0 & i \\ 6-i & i & 0 \end{bmatrix}$$

$$A^{\theta} = -\begin{bmatrix} 2i & 5+i & 6+i \\ -5+i & 0 & -i \\ -6+i & -i & 0 \end{bmatrix}$$

$\therefore$ *Hence* $A^{\theta} = -A$

$\therefore$ The matrix A is Skew Hermitian Matrix.

Note:

Let A be a square matrix expressed as B+iC where B and C are Hermitian and Skew Hermitian Matrices respectively.

$$A = \left[\frac{1}{2}\left(A + A^{\theta}\right)\right] + i\left[\frac{1}{2i}\left(A - A^{\theta}\right)\right] = B + iC$$

$$B = \frac{1}{2}\left(A + A^{\theta}\right) \text{ and } C = \frac{1}{2i}\left(A - A^{\theta}\right)$$

Unitary Matrix:

A square matrix A is said to be unitary matrix if $A^{\theta}A = 1$

Example 12:

Show that the matrix $A = \frac{1}{\sqrt{15}}\begin{bmatrix} -1+3i & -2-i \\ 1-2i & -3-i \end{bmatrix}$ is Unitary matrix.

Solution:

Here $A = \frac{1}{\sqrt{15}}\begin{bmatrix} -1+3i & -2-i \\ 1-2i & -3-i \end{bmatrix}$

$$A^{\theta} = \frac{1}{\sqrt{15}}\begin{bmatrix} -1-3i & 1+2i \\ -2+i & -3+i \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{15}\begin{bmatrix} -1+3i & -2-i \\ 1-2i & -3-i \end{bmatrix}\begin{bmatrix} -1-3i & 1+2i \\ -2+i & -3+i \end{bmatrix}$$

$$=\frac{1}{15}\begin{bmatrix}15 & 0\\ 0 & 15\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$\therefore AA^{\theta}=I$

$\therefore$ Hence A is Unitary Matrix.

Example 13:

Express the matrix, $A=\begin{vmatrix}2+i & 1 & 3-3i\\ i & 1-i & 2+i\\ 1+i & -3 & 5\end{vmatrix}$ As the Hermitian Matrix and Skew Hermitian Matrix.

Solution:

Let $A=\begin{vmatrix}2+i & 1 & 3-3i\\ i & 1-i & 2+i\\ 1+i & -3 & 5\end{vmatrix}......(I)$

$$\bar{A}=\begin{vmatrix}2-i & 1 & 3+3i\\ -i & 1+i & 2-i\\ 1-i & -3 & 5\end{vmatrix}$$

$$A^{\theta}=\begin{vmatrix}2+i & -1 & 1-i\\ 1 & 1+i & -3\\ 3+3i & 2-i & 5\end{vmatrix}......(II)$$

Adding I and II we get

$$A+A^{\theta}=\begin{vmatrix}2+i & 1 & 3-3i\\ i & 1-i & 2+i\\ 1+i & -3 & 5\end{vmatrix}+\begin{vmatrix}2+i & -i & 1-i\\ 1 & 1+i & -3\\ 3+3i & 2-i & 5\end{vmatrix}$$

$$=\begin{vmatrix}4 & 1-i & 4-4i\\ i+1 & 2 & i-1\\ 4+4i & -i-1 & 10\end{vmatrix}$$

$$B=\frac{1}{2}\left(A+A^{\theta}\right)=\frac{1}{2}\begin{vmatrix}4 & 1-i & 4-4i\\ i+1 & 2 & i-1\\ 4+4i & -i-1 & 10\end{vmatrix}......(III)$$

also $\left(A-A^{\theta}\right)=\begin{vmatrix}2+i & 1 & 3-3i\\ i & 1-i & 2+i\\ 1+i & -3 & 5\end{vmatrix}-\begin{vmatrix}2+i & -i & 1-i\\ 1 & 1+i & -3\\ 3+3i & 2-i & 5\end{vmatrix}$

$$=\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}$$

$$\frac{1}{2}\left(A-A^{\theta}\right)=\frac{1}{2}\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}......(IV)$$

Now, A=B+iC

$$A=\frac{1}{2}\begin{vmatrix} 4 & 1-i & 4-4i \\ i+1 & 2 & i-1 \\ 4+4i & -i-1 & 10 \end{vmatrix}+\frac{1}{2}\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}$$

Example 14:

Prove that the matrix, $A=\frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix}$

Solution:

Let $A=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$A^{\theta}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$A^{\theta}A=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\times\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$=\frac{1}{2}\begin{bmatrix} 1-i^2 & i-i \\ -i+i & -i^2+1 \end{bmatrix}$

$=\frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$

$\therefore AA^{\theta}=I$

Hence A is Unitary.

Check Your Progress:

(1) Show that the following matrices are Skew –Hermitian.

(i) $A=\begin{bmatrix} 2i & 2 & -3 \\ -2 & 4i & -6 \\ 3 & 6 & 0 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 4i & 1+i & 2+2i \\ i-1 & i & 5i \\ 2-2i & -5i & 3i \end{bmatrix}$

(2) Show that the following matrices are Unitary matrices.

(i) $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ i-1 & -1\end{bmatrix}$ (ii) $A=\frac{1}{2}\begin{bmatrix}1+i & -1+i\\ i+1 & 1-i\end{bmatrix}$

(3) If A is Hermitian matrix, then show that iA is Skew- Hermitian matrix.

4.8 LET US SUM UP

In this chapter we have learn

- ❖ Cayley Hamilton theorem & it application like Higher power of matrix & Inverse of matrix.
- ❖ Minimal .polynomial & derogatory & non-derogatory matrix.
- ❖ Complex matrix.
- ❖ Hermitian matrix. i.e $A=A^{\theta}$
- ❖ Skew Hermitian matrix. i.e $A^{\theta}=-A$
- ❖ Unitary matrix= $AA^{\theta}=I$.

4.9 UNIT END EXERCISE

1. Show that the given matrix A satisfies its characteristics equation.

i) $A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$

ii) $A=\begin{bmatrix}2 & 4 & 3\\ 0 & -1 & 1\\ 2 & 2 & -1\end{bmatrix}$

iii) $A=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}$

2. Using Cayley Hermitian theorem find inverse of the matrix A.

i) $A=\begin{bmatrix}-2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

ii) $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{bmatrix}$

3. Calculate A^5 by Cayley Hamilton Theorem if $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

4. Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{bmatrix}$. Find a similarity transformation that diagonalises matrix A.

5. Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ Find matrix P such that is diagonal matrix.

6. Diagonalise the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

7. For the matrix $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$.

Determine a matrix P such that is diagonal matrix.

8. If show that is Hermitian matrix.

9. Show that the following matrix are skew Hermitian matrix.

i) $A = \begin{bmatrix} 2i & -3 & 4 \\ 3 & 3i & -5 \\ -4 & 5 & 4i \end{bmatrix}$

ii) $= \begin{bmatrix} 0 & 1-i & 2+3i \\ -1-i & 0 & 6i \\ -2+3i & 6i & 0 \end{bmatrix}$

10. Show that the following matrix are unitary matrix

i) $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

ii) $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}$

11. Prove that a real matrix is unitary if it is orthogonal.

12. Check whether the following matrix is derogatory or non-derogatory.

i) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

iv) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

v) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

vi) $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

vii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

viii) $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

ix) $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

13. Show that the following matrix is derogatory also find minimal polynomial.

i) $$A=\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

ii) $$A=\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

iii) $$A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

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5

VECTOR CALCULAS

UNIT STRUCTURE

5.0 OBJECTIVES

After going through this unit, you will be able to

- Learn vector differentiation.
- Operators, del, grad and curl.
- Properties of operators

5.1 INTRODUCTION

Vector algebra deals with addition, subtraction and multiplication of vertex. In vector calculus we shall study differentiation of vectors functions, gradient, divergence and curl.

Vector:
Vector is a physical quantity which required magnitude and direction both.

Unit Vector:

Unit Vector is a vector which has magnitude 1. Unit vectors along co-ordinate axis are $\hat{i}$ and $\hat{j}$, $\hat{k}$ respectively.

$$\left|\hat{i}\right| = \left|\hat{j}\right| = \left|\hat{k}\right| = 1$$

Scalar Triple Vector:

Scalar triple product of three vectors is defined as $\bar{a}.\left(\bar{b}\times\bar{c}\right)$ or $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$. Geometrical meaning of $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ is volume of parallelepiped with cotter minus edges $\bar{a}$, $\bar{b}$ and $\bar{c}$.

We have,

$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\bar{b}\ \bar{c}\ \bar{a}\right] = \left[\bar{c}\ \bar{a}\ \bar{b}\right]$$
$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = -\left[\bar{b}\ \bar{a}\ \bar{c}\right]$$

Vector Triple Product:

Vector triple product of $\bar{a}\ \bar{b}$ and $\bar{c}$ is cross product of $\bar{a}$ and $\left(\bar{b}\times\bar{c}\right)$ i.e. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ or cross product of $\left(\bar{a}\times\bar{b}\right)$ and $\bar{c}$

$$\therefore\ \bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\,.\,\bar{c}\right)\bar{b} - \left(\bar{a}\,.\,\bar{b}\right)\bar{c}$$
$$\left(\bar{a}\times\bar{b}\right)\times\bar{c} = \left(\bar{a}\,.\,\bar{c}\right)\bar{b} - \left(\bar{b}\,.\,\bar{c}\right)\bar{a}$$

Remark : Vector triple product is not associative in general

i.e. $\therefore\ \bar{a}\times\left(\bar{b}\times\bar{c}\right) \neq \left(\bar{a}\times\bar{b}\right)\times\bar{c}$

Coplanar Vectors:

Three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar if $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$ for $\left|\bar{a}\right| \neq 0, \left|\bar{b}\right| \neq 0, \left|\bar{c}\right| \neq 0$

5.2 VECTORS DIFFERENTIATION

Let $\bar{v}$ be a vector function of a scalar t. Let $\partial\bar{v}$ be the small increment in a corresponding to the increment ∂t in t.

Then,

$$\partial\overline{v} = \overline{v}\left(t + \partial t\right) - \overline{v}(t)$$

$$\frac{\partial\overline{v}}{\partial t} = \frac{\overline{v}\left(t + \partial t\right) - \overline{v}(t)}{\partial t}$$

Taking limit $\partial t \longrightarrow 0$ we get,

$$\lim_{\partial t\to 0} \frac{\partial\overline{v}}{\partial t} = \lim_{\partial t\to 0} \frac{\overline{v}\left(t + \partial t\right) - \overline{v}(t)}{\partial t}$$

$$\frac{d\overline{v}}{dt} = \lim_{\partial t\to 0} \frac{\partial\overline{v}}{\partial t} = \lim_{\partial t\to 0} \frac{\overline{v}\left(t + \partial t\right) - \overline{v}(t)}{\partial t}$$

$$\frac{d\overline{v}}{dt} = \lim_{\partial t\to 0} \frac{\overline{v}\left(t + \partial t\right) - \overline{v}(t)}{\partial t}$$

Formulas of vector differentiation:

(i) $\frac{d}{dt} = \left(k\,\overline{v}\right) = k\frac{d\overline{v}}{dt}\left[\because \text{ k is a constant}\right]$

(ii) $\frac{d}{dt}\left(\overline{u} + \overline{v}\right) = \frac{d\overline{u}}{dt} + \frac{d\overline{v}}{dt}$

(iii) $\frac{d}{dt}\left(\overline{u}\,.\,\overline{v}\right) = \overline{u}\,.\,\frac{d\overline{v}}{dt} + \overline{v}\,.\frac{d\overline{u}}{dt}$

(iv) $\frac{d}{dt}\left(\overline{u}\times\overline{v}\right) = \overline{u}\times\frac{d\overline{v}}{dt} + \frac{d\overline{u}}{dt}\times\overline{v}$

(v) If $\overline{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Then, $\frac{d\overline{v}}{dt} = \frac{dv_1}{dt}\hat{i} + \frac{dv_2}{dt}\hat{j} + \frac{dv_3}{dt}\hat{k}$

Note:

If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $r = \left|\overline{r}\right| = \sqrt{x^2 + y^2 + z^2}$

Example 1:

If $\overline{r} = \left(t + 1\right)\hat{i} + \left(t^2 + t - 1\right)\hat{j} + \left(t^2 - t + 1\right)\hat{k}$ find $\frac{d\overline{r}}{dt}$ and $\frac{d^2\overline{r}}{dt}$

Solution:–

$\bar{r} = (t+1)\,\hat{i} + (t^2 + t - 1)\,\hat{j} + (t^2 - t + 1)\,\hat{k}$

$\frac{d\bar{r}}{dt} = \hat{i} + (2t+1)\,\hat{j} + (2t-1)\,\hat{k}$

$\frac{d^2\bar{r}}{dt^2} = 2\hat{j} + 2\,\hat{k}$

Example 2:

If $\bar{r} = \bar{a}\cos wt + \bar{b}\sin wt$ where w is constant show that

$\bar{r} \times \frac{d\bar{r}}{dt} = w\left(\bar{a} \times \bar{b}\right)$ and $\frac{d^2\bar{r}}{dt^2} = -w\,\bar{r}$

Solution: –

$\bar{r} = \bar{a}\cos wt + \bar{b}\sin wt$------------ (i)

$\frac{d\bar{r}}{dt} = \bar{a}\cos wt + \bar{b}\sin wt$------------ (ii)

$\therefore\ \bar{r} \times \frac{d\bar{r}}{dt} = \left(\bar{a}\cos wt + \bar{b}\sin wt\right) \times \left(-\bar{a}w\sin wt + \bar{b}w\cos wt\right)$

$= \left(\bar{a} \times \bar{b}\right) w\cos^2 wt - \left(\bar{b} \times \bar{a}\right) w\sin^2 wt \qquad \left[\because \begin{array}{l} \bar{a} \times \bar{a} = \bar{0} \\ \bar{b} \times \bar{b} = \bar{0} \end{array}\right]$

$= \left(\bar{a} \times \bar{b}\right) w\cos^2 wt + \left(\bar{a} \times \bar{b}\right) w\sin^2 wt \qquad \left[\because \begin{array}{l} \bar{b} \times \bar{a} = \bar{0} \\ = -\bar{a} \times \bar{b} \end{array}\right]$

$= \left(\bar{a} \times \bar{b}\right) w\left[\cos^2 wt + \sin^2 wt\right]$

$= \left(\bar{a} \times \bar{b}\right) w\,(1)$

$= w\left(\bar{a} \times \bar{b}\right)$

Again differentiating eq^n (ii) w.r.t. ‘t’

$\frac{d^2\bar{r}}{dt^2} = -\bar{a}\,w^2\cos wt - \bar{b}\,w^2\sin wt$

$= -w^2\left(\bar{a}\cos wt + \bar{b}\sin wt\right)$

$= -w^2 r$ from (i)

Example 3. Evaluate the following:

i) $\frac{d}{dt} = \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$ ii) $\frac{d}{dt} = \left[\bar{a} \quad \frac{d\bar{a}}{dt} \quad \frac{d^2\bar{a}}{dt^2}\right]$

Solution: – i) $\frac{d}{dt} = \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

$$= \frac{d}{dt}\left[\bar{a} \,.\, \left(\bar{b} \times \bar{c}\right)\right]$$

$$= \bar{a} \,.\, \frac{d}{dt}\left(\bar{b} \times \bar{c}\right) + \left(\bar{b} \times \bar{c}\right) . \frac{d\bar{a}}{dt}$$

$$= \bar{a} \,.\left(\bar{b} \times \frac{d\bar{c}}{dt} + \frac{d\bar{b}}{dt} \times c\right) + \left(\bar{b} \times \bar{c}\right) . \frac{d\bar{a}}{dt}$$

$$= \bar{a} \,.\left(\bar{b} \times \frac{d\bar{c}}{dt}\right) + \bar{a} \,.\left(\frac{d\bar{b}}{dt} \times \bar{c}\right) + \left(\bar{b} \times \bar{c}\right) . \frac{d\bar{a}}{dt}$$

$$= \left[\bar{a} \;\; \bar{b} \;\; \frac{d\bar{c}}{dt}\right] + \left[\bar{a} \;\; \frac{d\bar{b}}{dt} \;\; \bar{c}\right] + \left[\bar{b} \;\; \bar{c} \;\; \frac{d\bar{a}}{dt}\right]$$

Solution: – ii) $\frac{d}{dt} = \left[\bar{a} \quad \frac{d\bar{a}}{dt} \quad \frac{d^2\bar{a}}{dt^2}\right]$

$$= \left[\bar{a} \;\; \frac{d\bar{a}}{dt} \;\; \frac{d^3\bar{a}}{dt^3}\right] + \left[\bar{a} \;\; \bar{c} \;\; \frac{d^2\bar{a}}{dt^2} \;\; \frac{d^2\bar{a}}{dt^2}\right] + \left[\frac{d\bar{a}}{dt} \;\; \frac{d^2\bar{a}}{dt^2} \;\; \frac{d\bar{a}}{dt}\right]$$

(From Result i)

$$= \left[\bar{a} \;\; \frac{d\bar{a}}{dt} \;\; \frac{d^3\bar{a}}{dt^3}\right] + 0 + 0$$

$$= \left[\bar{a} \,.\; \frac{d\bar{a}}{dt} \;\; \frac{d^3\bar{a}}{dt^3}\right]$$

Example 4. Evaluate the following: $\frac{d}{dt} = \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right]$

Solution: $\frac{d}{dt} = \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right]$

$$= \left(\bar{a} \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \frac{d}{dt}\left(\bar{a} \times \bar{b}\right) \times c$$

$$= \left(\bar{a} \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \left(\bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \times \bar{b}\right) \times c$$

$$= \left(\bar{a} \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \left(\bar{a} \times \frac{d\bar{b}}{dt}\right) \times c + \left(\frac{d\bar{a}}{dt} \times \bar{b}\right) \times c$$

Example 5. Show that $\hat{r} \times \frac{d\hat{r}}{dt} = \frac{\hat{r} \times \frac{d\hat{r}}{dt}}{r^2}$, where $\hat{r} = \frac{\bar{r}}{r}$

Solution : We have $\hat{r} = \frac{\bar{r}}{r}$

$$\therefore \quad \frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$$

$$= \frac{r\frac{d\bar{r}}{dt} - \bar{r}\frac{dr}{dt}}{r^2}$$

$$= \frac{1}{r}\frac{d\bar{r}}{dt} - \frac{r}{r^2}\frac{dr}{dt}$$

L.H.S. $\hat{r} = \frac{\bar{r}}{r}$

$$= \frac{\bar{r}}{r} \times \left(\frac{1}{r}\frac{d\bar{r}}{dt} - \frac{\bar{r}}{r^2}\frac{dr}{dt}\right)$$

$$= \frac{\bar{r}}{r} \times \frac{1}{r}\frac{d\bar{r}}{dt} - \frac{\bar{r}\times\bar{r}}{r^2}\frac{dr}{dt}$$

$$= \frac{\bar{r}}{r^2} \times \frac{dr}{dt} - \bar{0} \qquad [\because \bar{r}\times\bar{r} = 0]$$

$$= \frac{r \times \frac{d\bar{r}}{dt}}{r^2}$$

$= \text{R.H.S}$

Example 6. If $\bar{r} = t^3\, i + \left(2t^3 - \frac{1}{5t^2}\right) j$. Then show that $\bar{r} \times \frac{d\bar{r}}{dt} = \hat{k}$

Solution:

$$\bar{r} = t^3 i + \left(2t^3 - \frac{1}{5t^2}\right) j$$

$$\frac{d\bar{r}}{dt} = 3t^2 i + \left(6t^2 + \frac{2}{5t^3}\right) j$$

L.H.S.

$$r \times \frac{d\bar{r}}{dt} = \begin{vmatrix} i & j & k \\ t^3 & 2t^3 - \frac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \frac{2}{5t^3} & 0 \end{vmatrix}$$

$$= i\,(0) - j(0) + k\left[t^3\left(6t^2 + \frac{2}{5t^3}\right) - 3t^2\left(2t^3 - \frac{1}{5t^2}\right)\right]$$

$$= k\left[\left(6t^5 + \frac{2}{5} - 6t^5 + \frac{3}{5}\right)\right]$$

$$= \hat{k}$$

$$= \text{R. H. S.}$$

Example 7. If $\bar{r} = \bar{a}\,e^{mt} + \bar{b}\,e^{-mt}$. Show that $\frac{d^2\bar{r}}{dt^2} = n^2\bar{r}$

Solution:

$\bar{r} = \bar{a}\,e^{mt} + \bar{b}\,e^{-mt}$...(i)

$$\frac{d\bar{r}}{dt} = m\,\bar{a}\,e^{mt} - m\,\bar{b}\,e^{-mt}$$

$$\frac{d^2\bar{r}}{dt^2} = m^2\,\bar{a}\,e^{mt} + m^2\,\bar{b}\,e^{-mt}$$

$$= m^2\left(\bar{a}\,e^{mt} + \bar{b}\,e^{-mt}\right)$$

$$= m^2\,\bar{r} \qquad \text{(from (i))}$$

$$\frac{d^2\bar{r}}{dt^2} = m^2\,\bar{r}$$

Check your progress:

(1) If $\frac{d\bar{u}}{dt} = \bar{w}\times\bar{u}$ and $\frac{d\bar{v}}{dt} = \bar{w}\times\bar{v}$

Show that $\frac{d}{dt}\left(\bar{u}\times v\right) = \bar{w}\times\left(\bar{u}\times v\right)$

(2) If $\bar{r} = t^2 i + \left(3t^3 - t^2\right) j + \left(7t + 1\right)\hat{k}$ Find $\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}$

(3) If: $\bar{r} = t\,\hat{i} - t\,\hat{j} + \left(st - 1\right)\hat{k}$, Find $\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \left|\frac{d\bar{r}}{dt}\right|, \left|\frac{d^2\bar{r}}{dt^2}\right|$

(4) If $\bar{r} = \bar{e}^t\,i + \left(2\cos 3t\right) j + \left(7\sin 3t\right)\hat{j}$ Find $\frac{d^2\bar{r}}{dt^2}$ at $t = \frac{\pi}{2}$

(5) Show that: $\bar{a}\,.\,\frac{d\bar{a}}{dt} = a\,\frac{da}{dt}$ where $a = a_1 i + a_2 j + a_3 k$ and a is magnitude of $\bar{a}$.

5.3 VECTOR OPERATOR

The vector differential operator ∇ is defined as $\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$.

5.3.1 Gradient:

The gradient of a scalar function is denoted by grad ϕ or $\nabla\phi$ and is defined as $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}$. Note that grad ϕ is a vector quantity.

5.3.2 Geometric meaning of gradient:

The grad ϕ is a vector right angled to the surface, whose equation is $\phi\,(x, y,\ z) = c,$ where c is constant.

Hence for $\bar{r} = x\,i\ + y\,j\ + z\,k$ any point on surface $\therefore\ \nabla\phi\ .\ d\bar{r} = 0$

i.e. $\nabla\phi$ at is right angles to $d\bar{r}$ and $d\bar{r}$ lies on the tangent plane to the surface at $P(\bar{r})$.

$\therefore\ \nabla\phi\ \perp\ d\bar{r}$

Geometrically $\nabla\phi$ represents a vector normal to the surface $\phi\,(x,\ y, z)$ = constant.

Example 8: Find grad ϕ, where $\phi = x^2\ y^3\ e^z$

Solution: grad $\phi\ = \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right)\left(x^2\ y^3\ e^z\right)$

$$= \hat{i}\ \frac{\partial}{\partial x}\left(x^2\ y^3\ e^z\right) + \hat{j}\ \frac{\partial}{\partial y}\left(x^2\ y^3\ e^z\right) + \hat{k}\ \frac{\partial}{\partial z}\left(x^2\ y^3\ e^z\right)$$

$$= \hat{i}\left(2xy^3e^z\right) + \hat{j}\left(3x^2\ y^2\ e^z\right) + \hat{k}\left(x^2\ y^3\ e^z\right)$$

$$= x\,y^2e^z\left(2y\,\hat{i} + 3x\hat{j} + xy\hat{k}\right)$$

Example 9: If $\bar{r} = x\hat{i} + y\hat{j}\ +\ z\hat{k}$ find grad r

Solution:

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2 + y^2 + z^2}$

$$\text{Grad } r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= \hat{i}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} + \hat{j}\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} + \hat{k}\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}$$

$$= \hat{i}\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x) + \hat{j}\frac{1}{2\sqrt{x^2+y^2+z^2}}.(2y) + \hat{k}\frac{1}{2\sqrt{x^2+y^2+z^2}}(2z)$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$\therefore \text{ grad } r = \frac{\bar{r}}{r}$$

Example 10: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ find grad $\frac{1}{r}$

Solution:

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{2r}{2x} = \frac{x}{r}, \quad \frac{2r}{2y} = \frac{y}{r}, \quad \frac{2z}{2r} = \frac{z}{r}$$

$$\text{grad } \frac{1}{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[i\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial y}\right]$$

$$= \frac{-1}{r^2}\left(\hat{i}\ \frac{x}{r}+\hat{j}\ \frac{y}{r}+\hat{k}\ \frac{z}{r}\right)$$

$$= \frac{-1}{r^2}.\ \frac{-1}{r}\ \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{-1}{r^3}\ r$$

$$= \frac{-r}{r^3}$$

Example 11: If $\phi = 2x^3y - y^2z$ find grad ϕ at (1, -1, 2)

Solution:

$$\text{grad } \phi = \left(\hat{i}\ \frac{\partial}{\partial x}+\hat{j}\ \frac{\partial}{\partial y}+\hat{k}\ \frac{\partial}{\partial z}\right)\left(2x^3y - y^2\ z\right)$$

$$= \hat{i}\ \frac{\partial}{\partial x}\ \left(2x^3y - y^2z\right) + \hat{j}\ \frac{\partial}{\partial y}\ \left(2x^3y - y^2\ z\right) + \hat{k}\ \frac{\partial}{\partial z}\ \left(2x^3y - y^2\ z\right)$$

$$= \hat{i}\ \left(6x^2y\right) + \hat{j}\ \left(2x^3 - 2yz\right) + \hat{k}\ \left(- y^2\right)$$

$$= \hat{i}\ 6x^2y + \hat{j}\ \left(2x^3 - 2yz\right) - \hat{k}\ y^2$$

At (1, -1, and 2)

$$\text{grad } \phi = 6\ (1)^2(-1)\ i + j\ \left(2(1)^3 - 2(-1)(2)\right) - \hat{k}\ (-1)^2$$

$$= 6\ i + j\ (2+4) - \hat{k}$$

$$= -6\ i + 6j\ - \hat{k}$$

Example 12: Evaluate grad e^{r^2}, where $r^2 = x^2 + y^2 + z^2$

Solution : Grad $\left(e^{r^2}\right) = \left(\hat{i}\ \frac{\partial}{\partial x}+\hat{j}\ \frac{\partial}{\partial y}+\hat{k}\ \frac{\partial}{\partial z}\right) e^{r^2}$

$$= \hat{i}\ \frac{\partial}{\partial x}\ \left(e^{r^2}\right) + \hat{j}\ \frac{\partial}{\partial y}\ \left(e^{r^2}\right) + \hat{k}\ \frac{\partial}{\partial z}\ \left(e^{r^2}\right)$$

$$= \hat{i}\ e^{r^2}.\partial r\frac{\partial r}{\partial x}\ + \hat{j}\ e^{r^2}.\partial r\frac{\partial r}{\partial y}\ + \hat{k}\ e^{r^2}.\partial r\frac{\partial r}{\partial z}$$

$$= \hat{i}\ e^{r^2}.\partial r.\frac{x}{r}\ + \hat{j}\ e^{r^2}.\partial r\frac{y}{r}\ + \hat{k}\ e^{r^2}.\partial r\frac{z}{r}$$

$$= r\ e^{r^2}\left(x\hat{i}\ + y\,\hat{j}\ + z\,\hat{k}\ \right)$$

$$= r\ e^{r^2}\overline{r}$$

Example 13: Find grad r^n

Solution: grad $r^n = \nabla r^n$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) r^n$$

$$= \hat{i}\,\frac{\partial}{\partial x}\,r^n + \hat{j}\,\frac{\partial}{\partial y}\,r^n + \hat{k}\,\frac{\partial}{\partial z}\,r^n$$

$$= \hat{i}\,n\,r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,n\,r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\,n\,r^{n-1}\frac{\partial r}{\partial z}$$

$$= \hat{i}\,n\,r^{n-1}\frac{x}{r} + \hat{j}\,n\,r^{n-1}\frac{y}{r} + \hat{k}\,n\,r^{n-1}\frac{z}{r}$$

$$= \hat{i}\,n\,r^{n-2}x + \hat{j}\,n\,r^{n-2}y + \hat{k}\,n\,r^{n-2}z$$

$$= n\,r^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= n\,r^{n-2}r$$

Example 14: Find grad $\log\left(x^2 + y^2 + z^2\right)$

Solution:

grad $\log\left(x^2 + y^2 + z^2\right)$ = grad $\log r^2$ = grad $(2\log r)$ = 2 grad $(\log r)$

$$= 2\left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(\log r)$$

$$= 2\left(\hat{i}\,\frac{\partial}{\partial x}(\log r) + \hat{j}\,\frac{\partial}{\partial y}(\log r) + \hat{k}\,\frac{\partial}{\partial z}(\log r)\right)$$

$$= 2\left(\hat{i}\,\frac{1}{r}\frac{\partial r}{\partial x} + \hat{j}\,\frac{1}{r}\frac{\partial r}{\partial y} + \hat{k}\,\frac{1}{r}\frac{\partial r}{\partial z}\right)$$

$$= 2\left(\hat{i}\,\frac{1}{r}\frac{x}{r} + \hat{j}\,\frac{1}{r}\frac{y}{r} + \hat{k}\,\frac{1}{r}\frac{z}{r}\right)$$

$$= \frac{2}{r^2}\left(x\hat{i} + y\,\hat{j}\,\frac{1}{r}\frac{y}{r} + z\,\hat{k}\right)$$

$$= \frac{2\bar{r}}{r^2}$$

Example 15: Show that grad $\left(\frac{\bar{a}.\,\bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n} = \frac{n\left(\bar{a}.\,\bar{r}\right)}{r^{n+2}}\,r$ where

$\bar{r}$ = r i +yj + zk

Solution: let

$\bar{a} = a_1\ i + a_2\ j + a_3\ k$

$\therefore\ \bar{a}\ .\bar{r} = a_1\ x + a_2\ y + a_3$

$\therefore\ \text{grad}\left(\frac{\bar{a}.\ \bar{r}}{r^n}\right)$

$= \nabla\left(\frac{\bar{a}.\ \bar{r}}{r^n}\right)$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(\frac{a_1\ x + a_2\ y + a_3\ z}{r^n}\right)$$

now $\therefore\ i\frac{\partial}{\partial x}\left(\frac{a_1\ x + a_2\ y + a_3\ z}{r^n}\right)$

$$= i\left(\frac{r^n\ a_1 - (a_1\ x + a_2\ y + a_3\ z)\ nr^{n-1}\ \frac{\partial r}{\partial x}}{r^{2n}}\right)$$

$$= i\left(\frac{r^n\ a_1 - (a_1\ x + a_2\ y + a_3\ z)\ nr^{n-1}\ \frac{x}{r}}{r^{2n}}\right)$$

$$= i\left(\frac{r^n\ a_1 - (a_1\ x + a_2\ y + a_3\ z)\ nx^{n-1}\ r^{n-2}}{r^{2n}}\right)$$

similarly

$$= \hat{j}\ \frac{\partial}{\partial y}\left(\frac{(a_1\ x + a_2\ y + a_3\ z)}{r^n}\right)$$

$$= \hat{j}\left(\frac{r^n\ a_2 - (a_1\ x + a_2\ y + a_3\ z)\ ny\ r^{n-2}}{r^{2n}}\right)$$

and

$$= \hat{k}\ \frac{\partial}{\partial z}\left(\frac{(a_1\ x + a_2\ y + a_3\ z)}{r^n}\right)$$

$$= \hat{k}\left(\frac{r^n\ a_3 - (a_1\ x + a_2\ y + a_3\ z)\ nz\ r^{n-2}}{r^{2n}}\right)$$

$\therefore\ \text{grad}\left(\frac{\bar{a}\ .\ \bar{r}}{r^n}\right)$

$$= \frac{r^n\ \left(a_1\ \hat{i} + a_2\ \hat{j} + a_3\ \hat{k}\right) - \left(a_1\ x + a_2\ y + a_3\ z\right)\ nz\ r^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r^{2n}}$$

$$= \frac{\bar{a}r^n - n\, r^{n-2}\ \bar{r}\left(\bar{a}\,.\bar{r}\right)}{r^{2n}}$$

$$= \frac{\bar{a}r^n}{r^{2n}} - \frac{-\,n\left(\bar{a}\,.\bar{r}\right)r^{n-2}\ \bar{r}}{r^{2n}}$$

$$= \frac{\bar{a}r^n}{r^{2n}} - \frac{n\left(\bar{a}\,.\bar{r}\right)}{r^{n+2}}\ \bar{r}$$

$$= \frac{\bar{a}}{r^n} - \frac{n\left(\bar{a}\,.\bar{r}\right)}{r^{n+2}}\ \bar{r}$$

Check your progress:

(1) If $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $\bar{r} = |\,\bar{r}\,|$

Show that:

a) $\text{grad}\,(\log\ r) = \dfrac{\bar{r}}{r^2}$

b) $\text{grad}\ r^3 = 3\,r\,\bar{r}$

c) $\text{grad}\ f\,(r) = f^1\,(r)\ \dfrac{\bar{r}}{r}$

(2) If $\phi = 4x^2yz + 3xyz^2 - 5xyz$

Find grad ϕ at (3, 2, -1)

(3) Show that $\text{grad}\ r^3 = -3\,r^{-5}\ \bar{r}$

(4) If $F\,(x,\ y,\ z) = x^2 + y^2 + z^2$ Find ∇F at (1, 1, 1)

(5) Show that $\nabla f\,(r) \times \bar{r} = 0$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(6) Find unit vector normal to the surface $x^2 + y^2 + z^2 = 3a^2$ at (a, a, a)

[Hint :- Unit vector normal to surface ϕ i.e. $\dfrac{\nabla\phi}{|\nabla\phi|}$]

5.3.1 Divergence:

If v (x, y, z) = $v_1\hat{i} + v_2\ \hat{j} + v_3\ \hat{k}$ can be defined and differentiated at each point (x, y, z) in a region of space then divergence of v is defined as div v = $\nabla\ .\ \bar{v}$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . \left(v_1\ i + v_2\ j + v_3\ k \right)$$

$$= \frac{\partial}{\partial x}(v_1) + \frac{\partial}{\partial y}(v_2) + \frac{\partial}{\partial z}(v_3)$$

Example 16 If $\overline{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j} + \left(y^2 - 2xy\right)\hat{k}$, find $\overline{F}$

Solution: div $\overline{F} = \nabla . \overline{F}$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . \left\{ \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j} + \left(y^2 - 2xy\right)\hat{k} \right\}$$

$$= \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}\left(y^2 - 2xy\right)$$

$$= 2x + 2x + 0$$

$$= 4x$$

Example 17 Show that div $\overline{r} = 3$ where $\overline{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$

Solution: div $\overline{r}$

$$= \nabla . \overline{F}$$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

Example 18 For $\overline{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$ show that div $\left(r^n\ \overline{r}\right) = (n+3)\,r^n$ where $r = \left|\overline{r}\right|$

Solution: L.H.S. div $\left(r^n\ \overline{r}\right) = \nabla . \left(r^n\ r\right)$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . r^n \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \frac{\partial}{\partial x}\left(r^n\ x\right) + \frac{\partial}{\partial y}\left(r^n\ y\right) + \frac{\partial}{\partial z}\left(r^n\ z\right)$$

$$= r^n(1) + x\,nr^{n-1}\ \frac{\partial r}{\partial x} + r^n(1) + y\,nr^{n-1}\ \frac{\partial r}{\partial y} + r^n(1) + z\,nr^{n-1}\ \frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + + z\frac{\partial r}{\partial z}\right)$$

$$= 3r^n + nr^{n-1}\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)$$

$$= 3r^n + nr^{n-1}\frac{\left(x^2 + y^2 + z^2\right)}{r}$$

$$= 3r^n + nr^{n-1}\frac{r^2}{r}$$

$$= 3r^n + nr^n$$

$$= (3 + n)r^n$$

= R.H.S.

Example 19 Evaluate div $(\hat{r})$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: We have $\hat{r}^n = \dfrac{\bar{r}}{r}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$\therefore$ div $(\hat{r})$

$$= \nabla.\hat{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} + \frac{r(1) - y\frac{\partial r}{\partial y}}{r^2} + \frac{r(1) - z\frac{\partial r}{\partial z}}{r^2}$$

$$= \frac{r - x\left(\frac{x}{r}\right)}{r^2} + \frac{r - y\frac{y}{r}}{r^2} + \frac{r - z\frac{z}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3}$$

$$= \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3}$$

$$= \frac{3r^2 - \left(x^2 + y^2 + z^2\right)}{r^3}$$

$$= \frac{3r^2 - r^2}{r^3}$$

$$= \frac{2}{r}$$

Example 20 If $F = x^2\ y^3\ z^4$ Find div (grad F)

Solution: grad F

$= \nabla F$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right)\ \left(x^2\ y^3\ z^4\right)$$

$= 2xy^3z^4\ \hat{i} + 3y^2z^4\ \hat{j} + 4x^2y^3z^3\ \hat{k}$

$\therefore$ div (grad F)

$$= \nabla . \left(2xy^3z^4\ \hat{i} + 3y^2x^2z^4\ \hat{j} + 4x^2y^3z^3\ \hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(2xy^3z^4\right) + \frac{\partial}{\partial y}\left(3y^2\ x^2\ z^4\right) + \frac{\partial}{\partial z}\left(4\ x^2y^3\ z^3\right)$$

$$= 2xy^3z^4 + 6x^2y\ z^4 + 12\ x^2y^3\ z^2$$

Example 21 Find the value of div $\left(\bar{a} \times \bar{r}\right) r^n$ where $\bar{a}$ is a constant vector and $\bar{r} = x\hat{i} + y\ \hat{j} + z\ \hat{k}$

Solution: div $\left(\bar{a} \times \bar{r}\right) r^n$

$$= \hat{i}\ \frac{\partial}{\partial x} . \left\{\left(\bar{a} \times \bar{r}\right) r^n\right\}$$

$$= \sum \hat{i} . \left[\left\{\frac{\partial}{\partial x} . \left\{\left(\bar{a} \times \bar{r}\right)\right\}\right\} r^n + \left(\bar{a} \times \bar{r}\right)\ \frac{\partial}{\partial x}\ \left(r^n\right)\right]$$

$$= \sum \hat{i} . \left[\left(\bar{a} \times \frac{\partial \bar{r}}{\partial x}\right) r^n + \left(\bar{a} \times \bar{r}\right) n\ r^{n-1}\ \frac{\partial r}{\partial x}\right]$$

$$\left[\because\ \frac{\partial \bar{a}}{\partial x} = 0\right]$$

$$= \sum \hat{i} . \left[\left(\bar{a} \times \hat{i}\right) r^n + \left(\bar{a} \times \bar{r}\right) n\ r^{n-1}\ \frac{x}{r}\right]$$

$$= \sum \hat{i} . \left[\left(\bar{a} \times \hat{i}\right) r^n + nx\ r^{n-2}\ \left(\bar{a} \times \bar{r}\right)\right]$$

$$= \sum nr^{n-2}\left(x\ \hat{i}\right)\left(\bar{a} \times \bar{r}\right) \qquad \left[\because\ \hat{i}.\left(\bar{a} \times \hat{i}\right) = 0\right]$$

$$= nr^{n-2}\left(\bar{a} \times \bar{r}\right)\ \sum x\ \hat{i}$$

$$= nr^{n-2}\left(\bar{a} \times \bar{r}\right)\ \bar{r} \qquad \left[\because \sum x\hat{i} = \bar{r}\right]$$

$$= nr^{n-2}\left[\left(\bar{a} \times \bar{r}\right) . \bar{r}\right]$$

$$= nr^{n-2}\left(0\right)$$

$$= 0$$

5.3.4 Solenoidal Function: A vector function $\bar{F}$ is called Solenoidal if $\text{div}\,\bar{F} = 0$ at all points of the function.

5.3.5 Curl: The curl of a vector point function $\bar{F}$ is defined as curl $\bar{F} = \nabla \times \bar{F}$ if $F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$\because \text{ curl } \bar{F} = \nabla \times \bar{F}$$

$$= \left(\nabla \times \bar{F}\right)$$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

The curl of the linear velocity of any particle of rigid body is equal to twice the angular velocity of body.

i.e. if $\bar{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ be the angular velocity of any particle of the body with position vector defined as $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then linear velocity $\bar{v} = \bar{w} \times \bar{r}$.

Hence curl $\bar{v} = \nabla \times \bar{v}$

$$= \nabla \times \left(\bar{w} \times \bar{r}\right)$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[\hat{i}\,\left(w_2 z - w_3 y\right) - \hat{j}\,\left(w_1 z - w_3 x\right) + \hat{k}\,\left(w_1 y - w_2 x\right)\right]$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= \hat{i}\,(w_1 + w_1) - \hat{j}\,(-w_2 - w_2) + \hat{k}\,(w_3 + w_3)$$

$$= 2w_1\hat{i} + 2w_2\hat{j} + 2w_3\hat{k}$$

$$= 2\overline{w}$$

$$\therefore \text{ curl } \overline{v} = 2\overline{w}$$

5.3.6 Irrotational field:

A vector point function $\overline{F}$ is called irrotational if $\overline{F} = \overline{0}$ at all points of the function.

Example 22 Find curl (curl $\overline{F}$) If $\overline{F} = x^2\ y\ \hat{i}\ -2\ x\ z\ \hat{j} + 2\ y\ z\ \hat{k}$ at (1, 0, 2)

Solution: Curl $\overline{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xy & 2yz \end{vmatrix}$$

$$= (2z + 2x)\hat{i} + \left(-2z - x^2\right)\hat{k}$$

$$\therefore \text{ curl curl } (\overline{F}) = \nabla \times \left[(2z + 2x)i + 0j + \left(-2z - x^2\right)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 2x & 0 & -2z - x^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(-2z - x^2\right) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(-2z - x^2\right) - \frac{\partial}{\partial z}(2z + 2x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2z + 2x)\right]$$

$$= \hat{i}(0) - \hat{j}\,(-2x - 2) + \hat{k}(0)$$

$$= (2x + 2)\,\hat{j}$$

At (1, 0, 2)

$$\left(\text{curl } \bar{F}\right) = \left[2(1) + 2\right]\hat{j}$$
$$= 4\hat{j}$$

Example 23 Find curl $\bar{V}$ if $\bar{V} = \left(x^2 + yz\right)\hat{i} + \left(y^2+2x\right)\hat{j}\left(z^2+xy\right)\hat{k}$

Solution: curl $\bar{V}$

$$= \nabla \times \bar{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(z^2+xy\right) - \frac{\partial}{\partial z}\left(y^2+2x\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(z^2+xy\right) - \frac{\partial}{\partial z}\left(y^2+yz\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(y^2+2x\right) - \frac{\partial}{\partial y}\left(x^2 + yz\right)\right]$$

$$= \hat{i}\left(x - x\right) - \hat{j}\left(y - y\right) + \hat{k}\left(z - z\right)$$

$$= \bar{0}$$

Example 24 Evaluate curl $\bar{r}$ where if $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: Curl $\bar{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \bar{0}$$

Example 25 Evaluate curl $\left(\frac{\hat{r}}{r}\right)$ where if $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution:

$$\hat{r} = \left(\frac{\bar{r}}{r^2}\right)$$

$$\therefore \frac{\hat{r}}{r} = \frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}$$

$$\therefore \text{curl}\left(\frac{\hat{r}}{r}\right) = \nabla \times \left(\frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{x}{r^2}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)\right]$$

$$= \hat{i}\left[\frac{-2z}{r^3}\ \frac{2r}{2y} + \frac{2y}{r^3}\ \frac{2r}{2z}\right] + \text{-------------}+\text{---------------}$$

$$= \hat{i}\left[\frac{-2z}{r^3}\ \frac{y}{r} + \frac{2y}{r^3}\ \frac{z}{r}\right] + \text{-------------}+\text{---------------}$$

$$= \hat{i}\left[\left(\frac{2yz - 2yz}{r^3}\right) + \hat{j}\left(\frac{2zx - 2zx}{r^3}\right) + \hat{k}\left(\frac{2xy - 2xy}{r^3}\right)\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \overline{0}$$

Example 25 If $\overline{F} = x^2y\,\hat{i} + xz\hat{j} + 2yz\hat{k}$ find div (curl $\overline{F}$)

Solution: curl $\overline{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}\left(x^2 z\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x\,z) - \frac{\partial}{\partial y}\left(x^2 z\right)\right]$$

$$= \hat{i}\,(2z - x) - \hat{j}(0 - 0) + \hat{k}\,\left(z - x^2\right)$$

$$= (2z - x)\,\hat{i} + \left(z - x^2\right)\hat{k}$$

div (curl $\overline{F}$)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left[(2z - x)\ \hat{i} + \left(z - x^2\right)\ \hat{k}\right]$$

$$= \frac{\partial}{\partial x}\ (2z - x) + \frac{\partial}{\partial z}\ \left(z - x^2\right)$$

$$= -1 + 1$$

$$= 0$$

Example 27 If $\overline{F}$ = grad $(xy + yz + zx)$, find (curl $\overline{F}$).

Solution: $\overline{F}$ = grad $(xy + yz + zx)$

$$= \nabla\ (xy + yz + zx)$$

$$== \left[i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] (xy + yz + zx)$$

$$= \hat{i}\ \frac{\partial}{\partial x}\ (xy + yz + zx) + \hat{j}\ \frac{\partial}{\partial y}\ (xy + yz + zx) + \hat{k}\frac{\partial}{\partial z}(xy + yz + zx)$$

$$= \hat{i}\ (y + z) + \hat{j}(x + z) + \hat{k}\ (y + x)$$

$\therefore$ (curl $\overline{F}$)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & x + z & x + y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x + y) - \frac{\partial}{\partial z}\ (x + z)\right] - \hat{j}\left[\frac{\partial}{\partial x}\ (x + y) - \frac{\partial}{\partial z}\ (y + z)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\ (x + z) - \frac{\partial}{\partial y}\ (y + z)\right]$$

$$= \hat{i}(1 - 1) - \hat{j}\ (1 - 1) + \hat{k}\ (1 - 1)$$

$$= 0\ \hat{i} + 0\ \hat{j} + 0\ \hat{k}$$

$$= \overline{0}$$

5.4 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

(i) $\nabla\ (f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla .\left(\overline{A} \pm \overline{B}\right) = \nabla . \overline{A} \pm \nabla . \overline{B}$

(iii) $\nabla \times \left(\overline{A} \pm \overline{B}\right) = \nabla \times \overline{A} \pm \nabla . \overline{B}$

Proof:

(i) $\nabla\left(f \pm g\right) = \hat{i}\left(\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(f \pm g\right)$

$= \hat{i}\frac{\partial}{\partial x}\left(f \pm g\right) \pm \hat{j}\frac{\partial}{\partial y}\left(f \pm g\right) + \hat{k}\frac{\partial}{\partial z}\left(f \pm g\right)$

$= \left(\hat{i}\frac{\partial}{\partial x}f + \hat{j}\frac{\partial}{\partial y}f + \hat{k}\frac{\partial}{\partial z}\right) \pm \left(\hat{i}\frac{\partial}{\partial x}g + \hat{j}\frac{\partial}{\partial y}g + \frac{\partial}{\partial z}g\right)$

$= \nabla f \pm \nabla g$

(ii) Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\therefore \nabla\,.\left(\bar{A} \pm \bar{B}\right)$

$= \nabla\,.\left[\left(\bar{A}_1 \pm \bar{B}_1\right)\hat{i} + \left(\bar{A}_2 \pm \bar{B}_2\right)\hat{j} + \left(\bar{A}_3 \pm \bar{B}_3\right)\hat{k}\right]$

$= \frac{\partial}{\partial x}\left(\bar{A}_1 \pm \bar{B}_1\right) + \frac{\partial}{\partial y}\left(\bar{A}_2 \pm \bar{B}_2\right) + \frac{\partial}{\partial z}\left(\bar{A}_3 \pm \bar{B}_3\right)$

$= \frac{\partial}{\partial x}\left(A_1\right) + \frac{\partial}{\partial y}\left(A_2\right) + \frac{\partial}{\partial z}\left(A_3\right) \pm \left[\frac{\partial}{\partial x}\left(B_1\right) + \frac{\partial}{\partial y}\left(B_2\right) + \frac{\partial}{\partial z}\left(B_3\right)\right]$

$= \nabla\,.\,\bar{A} \pm \nabla\,.\,\bar{B}$

(ii) Let

$\nabla \times \left(\bar{A} \pm \bar{B}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{A}_1 \pm \bar{B}_1 & \bar{A}_2 \pm \bar{B}_2 & \bar{A}_3 \pm \bar{B}_3 \end{vmatrix}$$

$= \sum \hat{i}\left[\frac{\partial}{\partial y}\left(\bar{A}_3 \pm \bar{B}_3\right) - \frac{\partial}{\partial z}\left(\bar{A}_2 \pm \bar{B}_2\right)\right]$

$= \sum \hat{i} \times \frac{\partial}{\partial x}\left(\bar{A} \pm \bar{B}\right)$

$= \sum \hat{i} \times \left(\frac{\partial \bar{A}}{\partial x} \pm \frac{\partial \bar{B}}{\partial x}\right)$

$= \sum \hat{i} \times \frac{\partial \bar{A}}{\partial x} \pm \sum \hat{i} \times \frac{\partial \bar{B}}{\partial x}$

$= \nabla \times \bar{A} \pm \nabla \times \bar{B}$

Check Your Progress:

(1) If $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ Evaluate div $\left(\bar{A}\times\bar{r}\right)$

(2) Prove that

$$\text{div}\left(\frac{\log r}{r}\bar{r}\right) = \frac{1}{r}\left(1 + 2\log r\right)$$

(3) For $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$) show that the vector div $\left(\frac{\bar{r}}{r^3}\right)$ is both solenoidal and irrotational.

(4) Prove that div $\left(\bar{a}.\,\bar{r}\right)\,\bar{a} = |\bar{a}|^2$

(5) For $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ show that $\nabla.\left(\nabla r^n\right) = n\,(n+1)r^{n-2}$

(6) show that the vector $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ solenoidal.

(7) If $\bar{A} = (ax + 3y + 4z)\,\hat{i} + (x - 2y + 3z)\,\hat{j} + (3x + 2y - z)\hat{k}$ is solenoidal find value of a.

(7) Find the direction derivative of a scalar field $\phi = x^2\,y\,z$ at (4, -1, 2) in the direction of (3, 2, 1).

[**Hint :-** direction derivative of ϕ (x, y, z) along $\bar{a}$ is = $\bar{a}$. grad ϕ]

5.4 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

1) If $\bar{S}$ represents displacement vector, $\frac{d\bar{s}}{dt}$ represents velocity and $\frac{d^2\bar{s}}{dt^2}$ represents acceleration.

2) For $\frac{d\bar{s}}{dt}\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

grad f $= \nabla$ F

grad $\bar{F} = \nabla.\,\bar{F}$

curl $\bar{F} = \nabla \times \bar{F}$

3) grad F and curl $\bar{F}$ are vector quantities.

4) div $\bar{F}$ is scalar quantity.

5.5 LET US SUM UP

In this chapter we have learn

- ❖ Differentiation of vectors.
- ❖ Partial derivative of vectors.
- ❖ The vector differential Operator Del.(∇)
- ❖ Divergence of a vector function.
- ❖ Curl of a vector.
- ❖ Properties of divergence, gradient & curl.

5.6 UNIT END EXERCISE

1) If $A = x^2yi - 2xzj + xy^2k, \quad B = 3zi + 2yj - 2x^2k$

Find the value $\frac{\partial^2}{\partial y \partial x}(A \times B)G + (1,0,1)$

2) If $r = xi + yi + zk$ prove that $\left(\frac{1}{R}\right) = \frac{-1}{R^3} r.$

where $R = |r|$

3) Find the unit normal vector to the surface at the point(1,0,1).

4) Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ the point (1,-1,1) in the direction of (3,-1,1)

5) If $f = 3x^2y - xyj + 3y^2zk$ find div F curl F.

6) Show that the vector $f = (x+3y)i + (y-3z)j + (x-2z)k$ is solenoid.

7) Show that the vector $f = (3x^2y)i + (x^3 - 2yz^2)j + (3z^2 - 2y^2z)k$ is irrotational.

8) Show that div r =3
where $r = xi + yi + zk$

9) Show that for any vector F
Div (Curl F)=0

10) If $a = a_1i + a_2i + a_3k$ and $r = xi + yj + zk$
Find Curl (r x a)

*****

6
DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

6.1 OBJECTIVE

After going through this chapter you will able to

i. Define differential equation
ii. Order & degree of differential equation
iii. Formulate the differential equation

6.2 INTRODUCTION

We have already learned differential equation in XII^{th} . Hence we are going to discuss differential equation in brief. In this chapter we discuss only formulation of differential equation.

6.3 DIFFERENTIAL EQUATION

Definition:-

An equation involving independent and dependent variables and the differential coefficients or differentials is called a differential equation.

e.g. 1 $\frac{dy}{dx}=9$

x=independent variable

y= depedent variable

2 $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

3 $\frac{d^ny}{dx^n}+y=0$

These are all examples of differential equations.

The differential equation is said to be ordinary if it contains only one independent variable. All the examples of above are of ordinary differential equations.

Order and Degree of a Differential Equations:-

(i) Order:-

The order of the differential equations is the order of the highest orderal derivatives present in the function or equation.

If y = f (x) is a function, then

$\frac{dy}{dx}$ is the first order derivative,

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the second order derivative.

e.g 1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Order = 2

2) E = Ri + L $\frac{di}{dt}$

Order = 1

Degree:-

The degree of differential equation is the degree of the highest ordered derivative in the equation when it is made free from radicals and fractions.

e.g.

1 $\frac{d^2y}{dx^2} + k^2 y = 0$

order = 2, degree = 1

2 $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + Y = 0$

Order =2, degree =1

3 $y = \left(\frac{dy}{dx}\right)x + \frac{1}{\frac{dy}{dx}}$

Order=1, degree=2

4 $\sqrt[3]{\frac{dy^2}{dx^2}} = \sqrt{\frac{dy}{dx}}$

$\therefore \left(\frac{d^2y}{dx^2}\right)^{1/3} = \left(\frac{dy}{dx}\right)^{1/2}$

Cubing both sides

$$\therefore \frac{d^{2y}}{dx^2} = \left(\frac{\text{dy}}{\text{dx}}\right)^{3/2}$$

Squaring both sides

$$\therefore \left(\frac{d^{2y}}{dx^2}\right)^2 = \left(\frac{\text{dy}}{\text{dx}}\right)^3$$

Order=2, degree=2

Solved examples:

Example 1: Find the order and degree of the following

i) $$e = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Solution:

$$\therefore \ \text{e}\cdot\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

Squaring both sides

$$\therefore \ \text{e}^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$\therefore$ order = 2, degree = 2

ii) $$\frac{d}{dx}\left\{x\left(\frac{d^2y}{dx^3}\right)^3\right\} + \sin(xy) = e^x$$

Solution:

$$\left(\frac{d^{3y}}{dx^3}\right)^3 + \ \text{x}\cdot 3 \left(\frac{\text{d}^3y}{dx^3}\right)^2 \cdot \frac{d^4y}{dx^4} + \sin(xy) = e^x$$

$\therefore$ Order = 4, degree=1

iii) $$y = x\cdot\frac{dy}{dx} + \frac{5}{\frac{dy}{dx}}$$

Solution:

$$\therefore \ \text{y}\cdot\frac{\text{dy}}{dx} = x\cdot\left(\frac{dy}{dx}\right)^2 + 5$$

$\therefore$ Order =1, degree=2

iv) $$y = x\cdot\frac{dy}{dx} + 5\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Solution:

$$\therefore\ y - x\cdot\frac{dy}{dx} = 5\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Squaring on both sides

$$\left(y\text{-}x\cdot\frac{dy}{dx}\right)^2 = 25\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore\ y^2 - 2xy\cdot\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = 25\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$\therefore$ Order =1, degree=2

Check your progress:

1) $$\frac{\partial^2 u}{\partial x^2} + u\cdot\frac{\partial u}{\partial y}$$

Ans : order =2, degree=1

2) $$\left(\frac{d^3y}{dx^3}\right)^4 + 5\left(\frac{d^3y}{dx^3}\right)^7 + 7\left(\frac{d^2y}{dx^2}\right)^{11} + \frac{dy}{dx} + y = e^x$$

Ans : order=3, degree=7

3) $$\left(y^{11}\right)^3 + \left(y^1\right)^4 = e^x$$

Ans : order=2, degree=3

4**)** $$y^{11} + \frac{x}{y^{11}} = 1$$

Ans : order=2, degree=2

5) $$y^{11} = \sqrt{1+y^{12}}$$

Ans : order=2, degree=2

$$y^1 + x = \left(y - xy^1\right)^{-2}$$

Ans : order =1, degree=3

6.4 FORMATION OF DIFFERENTIAL EQUATION

Formation of differential equation involves elimination of arbitrary consonants, in the relation of the variables.

Consider

$$y = ax^2 -------------1$$

Where y= independent variable

x = dependent variable

Differentiating equation (1) with respect to x

we have $\therefore \ \frac{\mathrm{dy}}{\mathrm{dx}} = 2ax -----------(2)$

From equation (1) we have

$$a = \frac{y}{x^2}$$

Put value of a in equation (2), we have

$$\therefore \quad \frac{\mathrm{dy}}{dx} = 2 \cdot \frac{y}{x^{\not 2}} \cdot \not x$$

$$\therefore \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{2y}{x}$$

$$\therefore \ \mathrm{x} \cdot \frac{\mathrm{dy}}{\mathrm{dx}} = 2y$$

$$\therefore \ \mathrm{x} \cdot \frac{\mathrm{dy}}{\mathrm{dx}} - 2y = 0$$

This is the required differential equation

Note:-

To eliminate two arbitrary constants, three equations are required. To eliminate three arbitrary constants, four equations are required.

In general to eliminate n arbitrary constants. (n+1) equations are required.

In other words elimination of n arbitrary consonants will bring us to differential equation of n^{th} order.

Solved Examples:-

Example 2: Form the differential equations if $y = c_1 \cos x + c_2 \sin x$

Solution: We have

Y= c_1 cos x + c_2 sinx -------------(1)

This equation contains two arbitrary constants, therefore we shall require three equations to eliminate c_1 and c_2 .

Differentiating equation (1) with respect to x

$$\therefore \ \frac{\mathrm{dy}}{\mathrm{dx}} = -c_1 \cos x + c_2 \cos x.$$

Again differentiate with respect to x

$$\therefore \ \frac{\mathrm{d}^2 y}{\mathrm{dx}^2} = -c_1 \cos x - c_2 \sin x$$

$$\frac{d^2y}{dx^2} = -(c_1 \cos x + c_2 \sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -y----------\left[\textit{from eq} \text{ ----(1)}\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

This is the required differential equation.

Example 3: Form the differential equation from

x= a sin (wt+c) where a and c are arbitrary constants.

Solution: We have,

x= a sin (wt+c)--------(1)

Differentiate equation (1) with respect

$$\therefore \quad \frac{dx}{dt} = +\text{ acos (cot+c)} \cdot \text{ w}$$

$$\therefore \quad \frac{dx}{dt} = +\text{ aw} \cdot \text{ cos(cot+c)}$$

Again differentiating w.r.t. 't'

$$\frac{d^2x}{dt^2} = \text{-a wsin } (\cot + c) \cdot w$$

$$\frac{d^2x}{dt^2} = -w^2 \left[\text{a sin } (\cot + c)\right]$$

$$\therefore \frac{d^2x}{dt^2} = -w^2x........\left[u \sin g \text{ equation 1}\right]$$

$$\therefore \frac{d^2x}{dt^2} + w^2x = 0$$

This is the required differential equation

Example 4: From the differential equation if y= log (ax)

Solution:

$$y = \log(ax)---------------(1)$$

Differentiate equation (1) with respect to x .

$$\therefore \quad \frac{dy}{dx} = \frac{1}{Ax} \cdot A$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \quad x \cdot \frac{dy}{dx} = 1$$

This is the required differential equation.

Example 5: Obtain the differential equation for the equation Y=cx+ c^2

Solution: we have,

$$y = cx + c^2 ------------(1)$$

Differentiate equation (1) with repect to x

$$\therefore \frac{dy}{dx} = c$$

Put value of c in equation (1)

$$\therefore y = x \cdot \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

This is the required differential equation.

Example 6: Obtain the differential equation for the relation

$\therefore$ y= $a \cdot e^{2x} + b \cdot e^{3x}$ *Where* a,b are constants.

Solution: we have,

$$\therefore y = a \cdot e^{2x} + b \cdot e^{3x} ---------------(1).$$

Here the number of arbitrary constants is two

Hence we shall require three equations to

Eliminate and b. So we differentiate the given equations twice.

$$\therefore \frac{dy}{dx} = 2a \cdot e^{2x} + 3b \cdot e^{3x} ---------------(2).$$

$$\therefore \frac{d^2y}{dx^2} = 4a \cdot e^{2x} + 9b \cdot e^{3x} ---------------(3)$$

From equation (1) (2) & (3) elimination of a & b gives directly

$$\begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 2 & 3 \\ \frac{d^2y}{dx^2} & 4 & 9 \end{vmatrix} = 0$$

In the determinant

1st column is LHS

Column 2nd column 2nd column contains coefficients $a \cdot e^{2x}$

Expanding the determinant

2nd column contains coefficients of $b \cdot e^{3x}$

$$y - (18 - 12) - \frac{dy}{dx}(9 - 4) + \frac{d^2y}{dx^2}(3 - 2) = 0$$

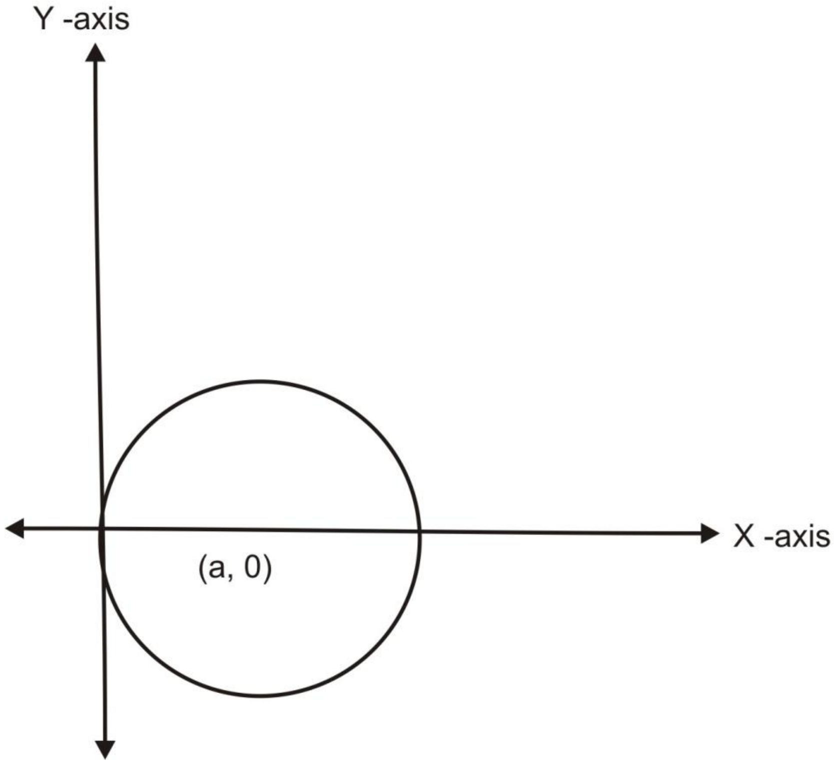
$$\therefore 6y - 5 \cdot \frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$$

$$\therefore \; \frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6y = 0$$

This is the required differential equation.

Example 7: Find the differential equation of all circles touching y axis at the origin and centers on x-axis

Solution:



The equation of such a circle is

$$\left(x-a\right)^2 + y^2 = a^2$$

i.e. $$x^2 - 2ax + \cancel{a}^2 + y^2 = \cancel{a}^2$$

$$\therefore x^2 + y^2 - 2ax = 0 ------------(1)$$

Where a is the only arbitrary contents
Differentiate equation (1) with respect to x We have

$$2x + 2y \cdot \frac{dy}{dx} - 2a = 0$$

$$x^2 + y^2 - 2x \cdot \left(x + y \cdot \frac{dy}{dx} \right) = 0$$

$$x^2 + y^2 - 2x^2 - 2xy \cdot \frac{dy}{dx} = 0$$

$$-x^2 + y^2 - 2xy \cdot \frac{dy}{dx} = 0$$

$\therefore\ 2xy \cdot \frac{dy}{dx} + \mathrm{x}^2 - y^2 = 0$

Which is the require differential equation.

Check Your Progress:

1) Form the differential equation of all circles of radius a.

Ans. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)$

2) Obtain the differential equation whose general solution is given by $y = e^x\left(A\cos x + B\sin x\right)$

Ans $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

3) Find the differential equation whose general solution is given by $y = c_1e^x + c_{2.}e^{-2x} + c_3 \cdot e^{3x}$

Ans $\frac{d\ ^3\ y}{d\ x^3} - 2\cdot\frac{d\ ^2y}{d\ x^2} - 5\frac{d\ y}{d\ x} - 6\ y = 0$

4) Obtain the differential equations for the following:

i) $y = A\cdot e^{3x} + B\cdot e^{2x}$

Ans $\frac{d^2y}{dx^2} - 5\cdot\frac{dy}{dx} + 6y = 0$

ii) $s = c_1e^{2t} + c_2 \cdot e^{-t}$

Ans $\frac{d^2s}{dt^2} - \frac{ds}{dt} - 2s = 0$

iii) $y = A$ cos 2t+ B sin 2t

Ans $\frac{\mathrm{d}^2y}{dt^2} + 4y = 0$

iv) $y = ax^3 + bx^2$

Ans $x^2\frac{d^2y}{dx^2} - 4x\cdot\frac{dy}{dx} + 6y = o$

v) $x = A\cos(nt + \alpha$

Ams $\frac{d^2y}{dt^2} + n^2x = 0$

vi $Y = A + Bx + Cx^2$

Soln $\frac{d^2y}{dx^2} = 0$

Vii $Y = \sin x + c$

Soln $\frac{dy}{dx} = \cos x$

Viii $y = (c_1 + c_2 x)e^x$

Ans $\frac{d^2y}{dx^2} - 2 \cdot \frac{dy}{dx} + y = 0$

6.5 LET US SUM UP

In this chapter we have learn

- ❖ Equation in term $\frac{dy}{dx}$ of is called differential equation.
- ❖ Degree & order of differential equation.
- ❖ Formation of differential equation while removing arbitrary constant likes A&B,&C.

6.6 UNIT END EXERCISE

1) Find the order 7 degree of Differential equation given below

i. $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{d^2y}{dx^2}\right) + 3\frac{dy}{dx} = y$

ii. $\left[i + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

iii. $\left(\frac{d^2y}{dx^2}\right)^{3/5} = \left(1 + \frac{dy}{dx}\right)$

iv. $2\frac{d^2y}{dx^2} + \sqrt[3]{1 + \left(\frac{dy}{dx}\right)^2} - y = 0$

v. $y = x\left(\frac{dy}{dx}\right) + \frac{1}{\left(dy/dx\right)}$

2) Formulate the differential equation

i. $Y = A + Blogx$

ii. $X = asin(w + +c)$

iii. $Y = c^x(Acosx + Bsinx)$

iv. $Y = e^{m\cos^{-1}x}$

v. $Y = ax^2 + bx$

vi. $Y = cx + 2c^2 + c^3$

vii. $X^2 + Y^2 = 2ax$

viii. $Y^2 = 4ax$

ix. $e^x + Ce^y = 1$

x. $Y = ACos2x + BSin2x$

*****

7

SOLUTION OF DIFFERENTIAL EQUATION

UNIT STRUCTURE

7.1 OBJECTIVES

After going through this chapter you will able to

- ❖ Find general & particular solution of differential equations.
- ❖ Classification of differential equation.
- ❖ Apply particular method first find the solution of differential **equation.**

7.2 INTRODUCTION

We have already formed differential equation in previous chapter. Here we are going to find solution of differential equation with different method. It is very useful in different field.

7.3 SOLUTION OF DIFFERENTIAL EQUATION

General Solutions:-

The general Solution of a differential equation is the most general relation between the dependent and the independent variable occurring in the equation which satisfies the given differential equation.

Particular Solutions:-

Any particular solution that satisfies the given equation is called a particular solution e.g.

$$\frac{dy}{dx} = 5$$

$$\therefore\ dy = 5dx$$

Integrating both sides we get

$$\therefore\ \int dy = 5 \cdot \int dx + constant$$

$$Y = 5x + C$$

This is called as general solution

Suppose C=7 is given

Then particular solution is given by putting of c in the general solution

$$\therefore\ y = 5x + 7$$

Check Point:-

1) Find the general solution and particular solution of the differential equation

$\frac{dy}{dx} = x$ When y = 4 at x = 0

Solution: $y = x^2/2 + c$

$$y = x^2/2 + 4$$

Differential equations of first order and of First Degree :-

An equation of the form,

$$M + N\frac{dy}{dx} = 0$$

Where 'M' and 'N' are functions of x and y or constant. is called differential equation of first order and first degree.

This equation can also be written as

$$Mdx + Ndy = 0$$

7.4 SOLUTION OF DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

There are many methods that can be used to solve the differential equations. Important one among those are listed below.

1) Variable seperable form.

2) Equations reducible to variable seperable form.

3) Homogeneous equations.

4) Exact differential equations.

5) linear differential equations.

6) Equations reducible to linear differential equation.(Bernoullis's differential equation)

7) Methods of substitution.

We will explain all these methods one by one in detail.

7.4.1 Variable Separable form:-

Working Rule

1) Consider the differential equation Mdx+ Ndy=0
2) If possible rearrange the terms and get f(x) dx +g(y).dy=0
3) Integrate and write constant of integration in suitable form, usually C.
4) Simplify if possible.

Solved Examples:-

Example 1: Solve $\left(3^x \tan y\right)\cdot dx+\left(1-e^x\right)Sec^2 y.dy=0$

Solution: $\left(3^x \tan y\right)\cdot dx+\left(1-e^x\right)Sec^2 y.dy=0$

$\div$ throughout by $\left(1\text{-e}^x\right)\cdot \tan y$ we get

$$\left(\frac{3\text{e}^x}{1\text{-e}^x}\right)dx+\frac{\sec^2 y}{\tan y}\cdot dy=0--------1)$$

This is in variable separable form

$\therefore$ Integrate equation (1), we get

$$\int\frac{3e^x}{1e^x}\ dx+\int\frac{Sec^2 y}{\tan y}\cdot dy=constant$$

$$\therefore \text{-}3\int\frac{\text{e}}{\text{e}^x-1}\cdot dx+\int\frac{Sec^2 y}{\tan y}\cdot dy=c$$

$$\therefore\ \text{-}3\log\left(\text{e}^x-1\right)+\log\ \ \tan y=\log c$$

$$\therefore\ \log\left(\text{e}^x-1\right)^{-3}+\log\ \ \tan y=\log c$$

$$\therefore\ log\left(e^x\text{ - }1\right)^{\text{-}3}\times tany=logc$$

$$\therefore\ \frac{tany}{\left(e^x\text{ - }1\right)^3}=c$$

$\because$ Removing log both side

$$\therefore\ tany=c\times\left(e^x\text{ - }1\right)^3$$

This is the general solution of a given differential equation.

Example 2: Solve $\frac{y}{x}\cdot\frac{dy}{dx}=\sqrt{1+x^2+y^2+x^2y^2}$

Solution: $$\frac{y}{x}\cdot\frac{dy}{dx}=\sqrt{\underline{1+x^2}+\underline{y^2\left(1+x^2\right)}}$$

$$\frac{y}{x}\times\frac{dy}{dx}=\sqrt{\left(1+x^2\right)\times\left(1+y^2\right)}$$

$$\frac{y}{x}\times\frac{dy}{dx}=\sqrt{\left(1+x^2\right)}\times\sqrt{\left(1+y^2\right)}$$

$$\therefore\ \frac{y}{\sqrt{\left(1+y^2\right)}}\times dy=x\sqrt{\left(1+x^2\right)}\times dx \text{----------} 1$$

This is in variable separable form

$\therefore$ Integrate equation (1)

$$\frac{1}{2}\cdot\int\frac{2y}{\sqrt{1+y^2}}\cdot dy=\frac{1}{2}\cdot\int 2x\cdot\sqrt{1+x^2}\cdot dx+c$$

$$\left\{\begin{array}{l}\backslash\int\frac{f^1(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c\\ \int\left[f(x)\right]^n\times f^1(x)dx\\ =\frac{\left[f(x)\right]^{n+1}}{n+1}\end{array}\right.$$

$$\therefore\ \frac{1}{2}\times\left[2\sqrt{1+y^2}\right]=\frac{1}{2}\times\left[\frac{2}{3}\times\left(1+x^2\right)^{3/2}\right]+c$$

$$\sqrt{1+y^2}=\frac{1}{3}\left(1+x^2\right)^{3/2}+c$$

This is in required general solution.

Example 3: Solve $(1+x)\cdot\frac{dy}{dx}+1=2e^{-y}$

Solution: The given equation is

$$\therefore\ (1+x)\cdot\frac{dy}{dx}=\left(2e^{-y}-1\right)$$

$$\therefore\frac{1}{\left(2e^{-y}-1\right)}\times dy=\frac{1}{x+1}\times dx$$

$$\therefore\frac{e^y}{e^y}\times\frac{1}{\left(2\times e^{-y}-1\right)}\times dy=\frac{1}{x+1}\times dx$$

$$\therefore\frac{e^y}{2-e^y}\cdot dy=\frac{1}{x+1}\cdot dx$$

$$\therefore\ 0=\frac{1}{x+1}\cdot dx-\frac{e^y}{2-e^y}\cdot dy$$

$$\therefore \frac{1}{x+1}\cdot dx+\frac{e^{y}}{e^{y-2}}\cdot dy=0-----(1)$$

This is in variable separable form,

Integrate equation (1), we get

$$\int\frac{1}{x+1}\cdot dx+\int\frac{e^{y}}{\left(e^{y}-2\right)}\ dy = \log c$$

$$\therefore \log\left(x+1\right)+\log\left(e^{y}-2\right)=\log c$$

$$\therefore\ \log\left[\left(x+1\right)\cdot\left(e^{y}-2\right)\right]=\log c$$

$$\therefore\left(x+1\right)\cdot\left(e^{y}-2\right)=c$$

This is the required general solution.

Example 4: Solve $3e^{x}\tan y\cdot dx+\left(1+e^{x}\right)\sec^{2}y\cdot dy=0$

given $y=\frac{\pi}{4}$ when x=0

Solution: The given equation is

$$3e^{x}\tan y\cdot dx+\left(1+e^{x}\right)\sec^{2}y\cdot dy=0$$

$\div$ through out by $\left(1+e^{x}\right)\cdot\tan y$

$$\therefore\ \frac{3e^{x}}{1+e^{x}}\cdot dx+\frac{\sec^{2}y}{\tan y}\cdot dy=0-----------(1)$$

This is in variable separable form,

Integrate equation (1) we get

$$\therefore\ \int\frac{3e^{x}}{1+e^{x}}\cdot dx+\int\frac{\sec^{2}y}{\tan y}\cdot dy=\log c$$

$$\therefore 3\int\frac{e^{x}}{1+e^{x}}\cdot dx+\int\frac{\sec^{2}y}{\tan y}\cdot dy=\log c\otimes$$

$$3\log\left(1+e^{x}\right)+\log\ \tan y=\log c$$

$$\therefore\ \log\left(1+e^{x}\right)^{3}+\log\ \tan y=\log c$$

$$\therefore\ \log\left[\left(1+e^{x}\right)^{3}\cdot\tan y\right]=\log c$$

$$\therefore\ \left(1+e^{x}\right)^{3}\cdot\tan y=c---------------(2)$$

This is the required general solution

To final particular solution:-

put y= $\pi/4$ at x=0 in equation --------(2)

$\therefore\ (1+1)^3 \cdot \tan\ \pi/4 = c$

$\therefore$ c=8

Put value of c in equation (2)

$$\therefore (1+e^x)^3 \cdot \tan y = 8$$

This is a particular solution

Example 5: Solve $\dfrac{dy}{dx} = \dfrac{x\ (2\log x\ +1)}{\sin\ y+\ y\cos y}$

Solution: The given equation is

$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$$

$$\therefore\ (\sin y + y\cos y)\cdot dy = x(2\log x+1)\cdot dx ------- (1)$$

This is in variable separable form

Integrate equation (1), we get

$$\int(\sin y + y\cos y)\cdot dy = \int x(2\log x+1)\cdot dx + cons\tan t$$

$$\therefore\ \int \text{siny}\cdot dy + \int y\cdot\cos y\cdot dy = 2\int x\cdot\log x\cdot dx + \int xdx + c$$

$$-\cos y + y\sin y + \cos y = 2\cdot\left[\log x\cdot\frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2}\right] + c$$

$$y\sin y = x^2\log x - x^2 + x^2 + c$$

$$\therefore\ y\sin y = x^2\log x + c$$

This is required general solution

Check Your Progress:

1) solve:

$$\frac{dy}{dx} = e^{x-y} + x^2\cdot e^{-y}$$

$$e^x + \frac{x^3}{3} - e^y = c$$

2) solve: $\left(y - x\cdot\dfrac{dy}{dx}\right) = a\cdot\left(y^2 + \dfrac{dy}{dx}\right)$

ans $(1-ay)(x+a) = cy$

3) solve: $\log\dfrac{dy}{dx} = ax + by$

ans $\dfrac{e^{ax}}{a} + \dfrac{e^{-by}}{b} = c$

4) solve: $x\cos x\cos y + \sin y\cdot\dfrac{dy}{dx} = 0$

ans $x\sin x + \cos x - \log\cos y = c$

5) solve: $Sec^2 x \cdot \tan y \cdot dx + \sec^2 y \cdot \tan x \cdot dy = 0$

ans $\tan x \cdot \tan y = c$

6) solve: $\frac{dy}{dx} = e^{x-2y}$

ans $\frac{1}{2} \cdot e^{2y} - e^{x} = c$

7.4.2 Equations Reducible to variable separable forms:

Sometimes we come across differential equations which cannot be converted into variable separable form by mere rearrangement of its terms.

These differential equation can be suitable substitution

Solved Examples:-

Example 6: solve: $\left(x-y\right)^2 \cdot \frac{dy}{dx} = a^2$

Solution: we have $\left(x-y\right)^2 \cdot \frac{dy}{dx} = a^2$ ----------(1)

Substitute x-y=t

Differentiating with respect to x, we get

$$1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = 1 - \frac{dt}{dx}$$

Using equation (1) we have

$$t^2 \cdot \left(1 - \frac{dt}{dx}\right) = a^2$$

$$\therefore 1 - \frac{dt}{dx} = \frac{a^2}{t^2}$$

$$\therefore \frac{dt}{dx} = 1 - \frac{a^2}{t^2}$$

$$\therefore \frac{dt}{dx} = \frac{t^2 - a^2}{t^2}$$

$$\therefore \frac{t^2}{t^2 - a^2} \cdot dt = dx$$

This is invariable separable form

Integrating we get

$$\int dx = \int \frac{t^2}{t^2 - a^2} \cdot dt + cons\tan t$$

$$\therefore \text{x}=\int \frac{t^2 - a^2 + a^2}{t^2 - a^2} \cdot dt + c$$

$$\therefore\ x=\int dt+\int \frac{a^2}{t^2-a^2}\cdot dt+c$$

$$\therefore\ x=t+\not{a}^2\cdot\frac{1}{2\not{a}}\cdot\log\left(\frac{t-a}{t+a}\right)+c$$

$$\therefore\ x=t+\frac{a}{2}\cdot\log\left(\frac{t-a}{t+a}\right)+c$$

t=x-y

$$\therefore\ x=x-y+\frac{a}{2}\cdot\log\left(\frac{x-y-a}{x-y+a}\right)+c$$

$$y=\frac{a}{2}\cdot\log\left(\frac{x-y-a}{x-y+a}\right)+c$$

This is the required general solution

Example 7: Solve $\frac{dy}{dx}=\cos(x+y)$

Solution: We have $\frac{dy}{dx}=\cos(x+y)$ ---------- (1)

Put $x+y=t$

Differentiating above with respect to x, we get

$$\therefore\ 1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{dt}{dx}-1$$

Using equation (1)

$$\therefore\ \frac{dt}{dx}-1=\cos t$$

$$\therefore\ \frac{dt}{dx}=1+\cos t$$

$$\therefore\ \frac{1}{1+\cos t}\cdot dt=dx$$

$$\therefore\ \frac{1}{2\cos^2 t/2}dt=dx$$

This is invariable separable form,

Integrating we get

$$\therefore\ \int\frac{1}{2\cos^2 t/2}\cdot dt=\int dx+cons\tan t$$

$$\therefore\ \frac{1}{2}\cdot\int\sec^2 t/2\cdot dt=x+c$$

$$\therefore \frac{1}{2}\cdot\frac{2}{1}\cdot\tan{}^{t}\!/_{2}=x+c$$

$$\therefore \tan{}^{t}\!/_{2}=x+c$$

t=x+y

$$\therefore \tan\left(\frac{x+y}{2}\right)=x+c$$

This is the required general solution,

Example 8: Solve $\left(4x+y\right)^{2}\cdot\frac{dx}{dy}=1$

Solution: The given equation is $\frac{\mathrm{dy}}{\mathrm{dx}}=\left(4x+y\right)^{2}$ -------(1)

Put $\left(4x+y\right)=t$

Differentiating above with respect to x

$$\therefore 4+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{dt}{dx}-4$$

Using equation (1), we have

$$\frac{dt}{dx}-4=t^{2}$$

$$\therefore \frac{dt}{dx}=t^{2}+4$$

$$\therefore \frac{1}{t^{2}+4}\cdot dt=dx$$

This is in variable separable form

Integrating we get,

$$\therefore \int\frac{1}{t^{2}+4}\cdot dt=\int dx+cons\tan t$$

$$\therefore \int\frac{1}{2}\cdot\tan^{-1}\left(\frac{t}{2}\right)=x+c$$

t=x+y

$$\therefore \frac{1}{2}\cdot\tan^{-1}\left(\frac{x+y}{2}\right)=x+c$$

$$\therefore \tan^{-1}\left(\frac{x+y}{2}\right)=2x+c_{1}\text{ where }\mathrm{c}_{1}=c$$

This is the required general solution

Example 9: Solve $\left(x+y\right)\cdot\frac{dy}{dx}+y=0$

Solution:

$$(x+y)\cdot\frac{dy}{dx}+y=0-----------(1)$$

Put $x+y=t$

Differentiating with respect to x, we get

$$\therefore\ 1+\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{dt}{dx}$$

$$\therefore\ \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{dt}{dx}-1$$

Using equation (1), we have

$$\therefore\ \mathrm{t}\cdot\left(\frac{dt}{dx}-1\right)+t-x=0$$

$$\frac{dt}{dx}-1=\frac{x-t}{t}$$

$$\therefore\ \frac{dt}{dx}-1=\frac{x}{t}-1$$

$$\therefore\ \frac{dt}{dx}=\frac{x}{t}$$

$$xdx=tdt$$

This is in variable separable form

Integrating we get,

$$\int \mathrm{xdx}=\int \mathrm{tdt}+\mathrm{constant}$$

$$\frac{x^2}{2}=\frac{t^2}{2}+c$$

$$\therefore\ \mathrm{x}^2=t^2+2c$$

t=x+y

$$\therefore\ \mathrm{x}^2=(x+y)^2+2c$$

$$\therefore\ \mathrm{x}^2=x^2+2xy+y^2+2c$$

$$\therefore\ 2\mathrm{xy}+y^2=-2c$$

$$\therefore\ y^2+2\mathrm{xy}=c_1\ \text{where } \mathrm{c}_1=-2c$$

This is the required general solution

Example 10: Solve $\left(\frac{y}{x}\cos\frac{y}{x}\right)\cdot dx-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot dy=0$

Solution:

The equation is, $\left(\frac{y}{x}\cos\frac{y}{x}\right)\cdot dx-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot dy=0$

Substitute $\frac{y}{x} = v$

$\therefore$ y=vx

Differentiating above with respect to x, we get

$$\therefore \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

But the above equation can be written as

$$\therefore \frac{y}{x} \cdot \cos\frac{y}{x} - \left(\frac{x}{y} \cdot \sin\frac{y}{x} + \cos\frac{y}{x}\right) \cdot \frac{dy}{dx} = 0$$

$$\therefore \text{v cosv-}\left(\frac{1}{v} \cdot \sin v + \cos v\right) \cdot \left(v + x \cdot \frac{dy}{dx}\right) = 0$$

By rearranging the terms, we have

$$\therefore \frac{1}{x} \cdot dx = -\frac{\sin v + v\cos v}{v\sin v} dv$$

$$\therefore \frac{1}{x} \cdot dx + \frac{\sin v + v\cos v}{v\sin v} dv = 0$$

This is in variable separable form

Integrating we get,

$$\therefore \int \frac{1}{x} \cdot dx + \int \frac{\sin v + v\cos v}{v\sin v} \cdot dv = cons\tan t$$

$$\therefore \text{logx+log}(\text{vsinv}) = c$$

$$\text{l}\phi\text{g}(\text{x} \cdot \text{vsinv}) = \text{l}\phi\text{g}\, c$$

$$\text{xv} \cdot \text{sinv=c}$$

$$\text{v=}\frac{y}{x}$$

$$\therefore \text{x} \cdot \frac{y}{x} \sin\frac{y}{x} = c$$

$$\therefore \text{ysin} \frac{y}{x} = c$$

This is the required general solution

Check Your Progress:

Solve the following

1) $\frac{dy}{dx} + e^{y/x} = \frac{y}{x}$ Ans : log cx=e$^{-y/x}$

2) $\left(1 + e^{x/y}\right) + e^{x/y}\left(1 - \frac{x}{y}\right) \cdot \frac{dy}{dx} = 0$ Ans : $\text{x+y} \cdot \text{e}^{x/y} = c$

3) $\left(2x-y\right)\cdot e^{y/x}+\left(y+x\cdot e^{y/x}\right)\cdot\frac{dy}{dx}=0$ Ans:$y^2+2x^2e^{y/x}=c$

$$\left[\tan\frac{y}{x}-\frac{y}{x}\cdot\sec^2\frac{y}{x}\right]dx+\sec^2\frac{y}{x}\cdot dy=0$$

Ans $x+\tan\left(\frac{y}{x}\right)=c$

7.4.3 Homogeneous Equations

A differential equation Mdx+Ndy=0 is said to be homogeneous if M & N are homogeneous functions of x and y of same degree

Working Rule:

1) Check whether differential equation is homogenous in x and y
2) Express $\frac{dy}{dx}$ in terms of x and y
3) Put y=vx
4) $\therefore\ \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$
5) Separate x and y variables and get F(ex)dx+ g(v) dv=0
6) Solve by integration
7) Put $v=\frac{y}{x}$ and simplify

Solved examples:-

Example 11: Solve $\left(x^2+y^2\right)dx+2xy\cdot dy=0$

Solution: We have $\left(x^2+y^2\right)dx+2xy\cdot dy=0$

Here M and N are homogeneous expressions in x and y of the second degree

$$\therefore\ 2xy\cdot dy=-\left(x^2+y^2\right)dx$$

$$\therefore\ 2xy\cdot dy=-\left(x^2+y^2\right)dx$$

$$\therefore\ \frac{dy}{dx}=-\cdot\left(\frac{x^2+y^2}{2xy}\right)------------(1)$$

put y=vx

$$\therefore\ \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

Using equation 1 we have

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{-2x\cdot vx}$$

$$\therefore\ v + x\frac{dv}{dx} = \frac{x^2\left(1+v^2\right)}{-2v\cdot x^2}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1+2v^2}{-2v} - 1$$

$$\therefore\ x\frac{dv}{dx} = \frac{1+3v^2}{-2v}$$

$$\therefore\ \frac{-2v}{1+3v^2}\cdot dv = \frac{1}{x}dx$$

This is in variable separable form

Integrating above expression we have

$$\therefore\ -\frac{1}{3}\int\frac{6v}{1+3v^2}\cdot dv = \int\frac{1}{x}\ dx + constant$$

$$\therefore\ -\frac{1}{3}\log\left(1+3v^2\right) = \log x + \log c$$

$$\therefore\ -\frac{1}{3}\log\left(1+3v^2\right) = \log\left(cx\right)$$

$$\therefore\ \log\left(1+3v^2\right) = -3\log\left(cx\right)$$

$$\therefore\ \log\left(1+3v^2\right) = -3\log\left(cx\right)^{-3}$$

$$\therefore\ 1+3v^2 = \frac{1}{c^3x^3}$$

$$v = \frac{y}{x}$$

$$\therefore\ 1+3\cdot\frac{y^2}{x^2} = \frac{1}{c^3x^3}$$

$$\therefore\ x^3 + 3xy^2 = \frac{1}{c^3}$$

$$\therefore\ x^3 + 3xy^2 = k \text{ where } k=\frac{1}{c^3}$$

This is the required general solution

Example 12: Solve $y^2 + x^2\cdot\frac{dy}{dx} = xy\cdot\frac{dy}{dx}$

Solution:

The given equation is $y^2 + x^2\cdot\frac{dy}{dx} = xy\cdot\frac{dy}{dx}$

$$\therefore\ y^2 = xy\cdot\frac{dy}{dx} - x^2\cdot\frac{dy}{dx}$$

$$\therefore\ y^2 = \frac{dy}{dx}\left(xy - x^2\right)$$

$$\therefore \ \frac{dy}{dx} = \frac{y^2}{xy - x^2} ---------(1)$$

This is a homogeneous equation

Put y=vx

$$\therefore \ \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

Using equation (1),we have

$$\therefore \ v + x \cdot \frac{dv}{dx} = \frac{v^2 x^2}{x \cdot vx - x^2}$$

$$\therefore \ v + x \cdot \frac{dv}{dx} = \frac{v^2 \not{x}^2}{\not{x}^2 (v-1)}$$

$$\therefore \ x \cdot \frac{dv}{dx} = \frac{v^2}{(v-1)} - v$$

$$\therefore \ x \cdot \frac{dv}{dx} = \frac{v}{(v-1)}$$

$$\therefore \ \frac{v-1}{v} \cdot dv = \frac{1}{x} \cdot dx$$

$$\therefore \ \left(1 - \frac{1}{v}\right) dv = \frac{1}{x} dx$$

This is in variable separable form

Integrating we get,

$$\int \left(1 - \frac{1}{v}\right) \text{dv} = \int \frac{1}{\text{x}} \ dx + constant$$

$$\therefore \ \text{vlogv=logx+logc}$$

$$\therefore \ \text{v=logv+logx+logc}$$

$$\therefore \ \text{v=log}\left(\text{vxc}\right)$$

$$\text{v=}\frac{\text{y}}{\text{x}}$$

$$\therefore \ \frac{\text{y}}{\text{x}} = \text{log}\left(\frac{\text{y}}{\not{\text{x}}} \cdot \not{x} \cdot c\right)$$

$$\therefore \ \frac{\text{y}}{\text{x}} = \text{log } c\text{y}$$

$$\therefore \ \text{y=xlog } c\text{y}$$

This is the required general solution

Example 13: solve $\left(x^3 + y^3\right) dx - 3xy^2 \cdot dy = 0$.

Solution:

$$\left(x^3+y^3\right)dx-3xy^2\cdot dy=0$$

This is a homogenous equation

$$\left(x^3+y^3\right)dx=3xy^2\cdot dy$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{x^3+y^3}{3xy^2}--------(1)$$

Put $y=vx$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=v+x\frac{dv}{dx}$$

Using equation 1 we have

$$\therefore v+x\frac{dv}{dx}=\frac{x^3+v^3x^3}{3x\cdot v^2x^2}$$

$$\therefore v+x\frac{dv}{dx}=\frac{x^3\left(1+v^3\right)}{3v^2\cdot x^3}$$

$$\therefore x\frac{dv}{dx}=\frac{1+v^3}{3v^2}-v$$

$$\therefore x\frac{dv}{dx}=\frac{1-2v^3}{3v^2}-$$

$$\therefore \frac{3v^2}{1-2v^3}\cdot dv=\frac{1}{x}dx$$

This is in variable separable form

Integrating we have

$$-\frac{1}{2}\cdot\int\frac{6v^2}{2v^3-1}\cdot dv=\int\frac{1}{x}\,dx+constant$$

$$\therefore -\frac{1}{2}\log\left(2v^3-1\right)=\log x+\log c$$

$$\therefore \log\left(2v^3-1\right)=-2\log\left(cx\right)$$

$$\therefore \cancel{\log}\left(2v^3-1\right)=\cancel{\log}\left(cx\right)^{-2}$$

$$\therefore \left(2v^3-1\right)=\frac{1}{c^2x^2}$$

Put $\mathrm{v}=\frac{\mathrm{y}}{\mathrm{x}}$

$$\therefore 2\frac{\mathrm{y}^3}{\mathrm{x}^3}-1=\frac{1}{c^2x^2}$$

$$\therefore 2\mathrm{y}^3-x^3=\frac{x}{c^2}$$

$$\therefore\ 2y^3 - x^3 = kx \text{ where } k=\frac{1}{c^2}$$

This is the required general solution

Example 14: solve $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\cdot dy = 0$

Solution:

The given equation is

$$\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\cdot dy = 0$$

$$\therefore\ \frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}}$$

$$\therefore\ \frac{dy}{dx} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}} \text{---------}(1)$$

This is a homogeneous equation

Put $y = vx$

$$\therefore\ \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

Using equation 1 we have

$$\therefore\ \not{v} + x\cdot\frac{dv}{dx} = \not{v} - \frac{\tan v}{\sec^2 v}$$

$$\therefore\ x\cdot\frac{dv}{dx} = \frac{\tan v}{\sec^2 v}$$

$$\therefore\ \frac{\sec^2 v}{\tan v}\cdot dv = -\frac{1}{x}\cdot dx$$

$$\therefore\ \frac{\sec^2 v}{\tan v}\cdot dv + -\frac{1}{x}\cdot dx = 0$$

This is in variable separable form

Integrating we get,

$$\therefore\ \int\frac{\sec^2 v}{\tan^2 v}\cdot dv + \int\frac{1}{x}dx = cons\tan t$$

$\therefore$ log tanv + log x = log c

$\therefore$ l$\not{o}$g (tanv $\cdot$ x) = l$\not{o}$g c

$\therefore$ $x\cdot\tan v = c$

Put $v = \frac{y}{x}$

$\therefore\ x \cdot \tan\frac{y}{x} = c$

This is the required general solution

Check Your Progress:

1) solve the following

i) $xdy - ydx = \sqrt{x^2 + y^2} \cdot dx$

ans $y + \sqrt{x^2 + y^2} = cx^2$

ii) $\left(x + y \cdot \cot\frac{x}{y}\right)dy - ydx = 0$

ans $y = c\sec\frac{x}{y}$

iii) $y^2 + x^2 \cdot \frac{dy}{dx} = xy \cdot \frac{dy}{dx}$

ans $cy = e^{y/x}$

iv) $\left(x^2 - y^2\right)dx = 2xydy$

ans $x\left(x^2 - 3y^2\right) = c$

v) $x\frac{dy}{dx} = y + \sqrt{x^2 + a^2}$

ans $y = c \cdot e^{x^2/3y^2}$

vi) $\left(x + y\right) \cdot \frac{dy}{dx} = x - y$

ans $-y^2 - 2xy + x^2 = c$

7.4.4 Exact Differential Equation

Definition:-

The equation Mdx+Ndy=0 is said to be an exact differential equation if and only it.

Mdx+ Ndy=du

Where u is some function of x and y

e.g. xdy+ydx=0 is exact

$\because$ u=xy

Where

xdy+ydy = du

Necessary and sufficient condition :-

The necessary and sufficient condition that the equation Mdx+Ndy=0 is exact is.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Rules for the General solution:-

If the equation Mdx+Ndy=0 is exact then its general solution is given by

$$\int M\ \left(treat\ \text{y as constant}\right)\ dx+\int N\ \left(terms\ f\text{ree from x}\ \right)\ dy=c$$

Where

(1) In first integral with respect to x, treat y as constant

(ii) In second integral do not take the terms containing x i.e. take only those terms of N which are free from x. If no such term is available then second integrals may not be considered.

(iii) c is arbitrary constant of Integration.

Solved Examples:-

Example15: Solve $\left(5x^4+6x^2y^2-8xy\right)\ dx+\left(4x^3y-12x^2y^2-5y^4\right)\cdot dy=0$

Solution: The given equation is:

$$\left(5x^4+6x^2y^2-8xy^3\right)dx+\left(4x^3y-12x^2y^2-5y^4\right)dy=0-----(1)$$

$$\therefore\ \text{M}=5x^4+6x^2y^2-8xy^3$$

$$\text{N}=4x^3y-12x^2y^2-5y^4$$

$$\therefore\ \frac{\partial\text{M}}{\partial\text{Y}}=\frac{\partial}{\partial\text{Y}}\left(5x^4+6x^2y^2-8xy^3\right)$$

$$=0+12x^2y-24xy^2$$

$$\therefore\ \frac{\partial\text{M}}{\partial\text{Y}}=12x^2y-24xy^2$$

$$\therefore\ \frac{\partial\text{N}}{\partial\text{x}}=\frac{\partial}{\partial\text{x}}\left(4x^3y-12x^2y^2-5y^4\right)$$

$$=\ 12x^2y-24xy^2-0$$

$$\therefore\ \frac{\partial\text{N}}{\partial\text{x}}=12x^2y-24xy^2$$

$$\therefore\ \frac{\partial M}{\partial\text{y}}=\frac{\partial N}{\partial\text{x}}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int M\left(treat\ \text{y constant}\right)dx+\int N\left(terms\ \text{free from x}\right)\cdot dy=c$$

$$\therefore \int\left(5x^4+6x^2y^2-8xy^3\right)dx+\int\left(-5y^4\right)\cdot dy=c$$

$$\therefore\ \cancel{5}\cdot\frac{x^5}{\cancel{5}}+\cancel{6}^2 y\cdot\frac{x^3}{\cancel{3}}-\cancel{8}^4 y^3\cdot\frac{x^2}{\cancel{2}}-\cancel{5}\cdot\frac{y^5}{\cancel{5}}=c$$

$$x^5+2x^3y^2-4x^2y^3-y^5=c$$

This is the required general solution

Example 16: Solve $\dfrac{dy}{dx}=-\dfrac{4x^3y^2+y\cos xy}{2x^4y+x\cos xy}$

Solution**:**

The given equation is

$$\frac{dy}{dx}=-\frac{4x^3y^3+y\cos xy}{2x^4y+x\cos xy}$$

$$\therefore\left(4x^3y^2+y\cos xy\right)\ dx+\left(2x^4+y\cos xy\right)\ dy=0\text{...........(1)}$$

Comparing with Mdx+Ndy=0; we have

$$M=4x^3y^2+y\cos xy$$

$$N=2x^4y+x\cos xy$$

$$\therefore\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(4x^3y^2+y\cos xy\right)$$

$$\frac{\partial M}{\partial y}=8x^3y^2+\cos xy-xy\sin xy$$

$$\therefore\frac{\partial N}{\partial x}=\frac{\partial}{x}\left(2x^4y\ \ +y\cos xy\right)$$

$$\frac{\partial N}{\partial x}=8x^3y\ \ +\cos xy-xy\sin xy$$

$$\therefore\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int Min\left(treat\ \text{y constant}\right)dx+\int N\left(terms\ \text{free from x}\right)\cdot dy=c$$

$$\left(4x^3y^2+y\cos xy\right)\ \text{dx}+\int\text{ody}=\text{c}$$

$$4y^2\int x^3dx+y\int\cos\ \text{xy}=\text{c}$$

$$4y^2\cdot\frac{x^4}{4}+y\frac{\sin xy}{y}=c$$

$$\therefore\quad x^4y^2+\sin xy=c$$

This is the required general solution

Example 17: Solve $\left(x-2e^y\right)dy+\left(y+x\sin x\right)dx=0$.

Solution:

The equation given is

$$\left(x-2e^{y}\right)dy+\left(y+x\sin x\right)dx=0$$

$$\therefore\ \left(\text{y+xsinx}\right)dx+\left(x-2e^{y}\right)dy=0-----(1)$$

Comparing with Mdx+Ndy=0; we have

$$M=y+x\sin x$$

$$N=x-2e^{y}$$

$$\therefore\ \frac{\partial \text{M}}{\partial \text{y}}=\frac{\partial}{\partial y}\left(y+\sin x\right)$$

$$\therefore\ \frac{\partial \text{M}}{\partial \text{y}}=1$$

$$\therefore\ \frac{\partial \text{N}}{\partial \text{x}}=\frac{\partial}{\partial x}\left(x-2e^{y}\right)$$

$$\therefore\ \frac{\partial \text{N}}{\partial \text{x}}=1$$

$$\therefore\ \frac{\partial \text{M}}{\partial \text{y}}=\frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int M\left(treat\ \text{y constant}\right)dx+\int N\left(terms\ \text{free from x}\right)\cdot dy=c$$

$$\therefore\ \int\left(\text{y+xsinx}\right)dx+\int\left(-2\cdot e^{y}\right)\cdot dy=c$$

$$\therefore xy+\left[x\left(-\cos x\right)+\sin x\right]-2\cdot e^{y}=c$$

This is the required general solution

Example 18: Solve

$$\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)\text{ dy}=0$$

Solution: The given equation is

$$\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)\cdot dy=0----(1)$$

Comparing with Mdx+Ndy=0; we have

$$M=y\left(1+\frac{1}{x}\right)+\cos y$$

$$N=x+\log x-x\sin y$$

$$\therefore\ \frac{\partial \text{M}}{\partial \text{Y}}=\frac{\partial}{\partial y}\left(y\left(1+\frac{1}{x}\right)+\cos y\right)$$

$$\frac{\partial M}{\partial Y} = 1 + \frac{1}{x} - \sin y$$

$$\therefore \ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \cdot \left(x + \log x - x \sin y\right)$$

$$\therefore \ \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial x}$$

Hence the differential equation (1) is exact

Its solution is given by

$$\int M\left(treat \text{ y constant}\right) \text{dx} + \int \text{N} \left(\text{terms free from x}\right) \text{dy} = c$$

$$\int \left(y \cdot \left(1 + \frac{1}{x}\right) + \cos y \right) \text{dx} + \int \text{ody} = \text{c}$$

$$y \cdot \int \left(1 + \frac{1}{x}\right) \text{dx} + \int \cos \text{y dy} = \text{c}$$

$$\text{y}\left(\text{x} + \text{logx}\right) + x \cos y = c$$

This is the required general solution

Example 19: Solve $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$

Solution: The given equation is

$$\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$$

$$\therefore \ \frac{dy}{dx} = -\frac{\left(y \cos x + \sin y + y\right)}{\left(\sin x + x \cos y + x\right)}$$

$$\therefore \ \left(\sin x + x \cos y + x\right) dy = -\left(y \cos x + \sin y + y\right) \text{dx}$$

$$\therefore \ \left(y \cos x + \sin y + y\right) \text{dx} + \left(\sin x + x \cos y + x\right) dy = 0 ----(1)$$

Comparing with Mdx+Ndy=0; we have

$$\text{M} = y \cos x + \sin y + y$$

$$\text{N} = \sin x + x \cos y + x$$

$$\therefore \ \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y \cos x + \sin y + y\right)$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\therefore \ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\sin x + x \cos y + x\right)$$

$$\therefore \ \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the differential equation (1) is exact

Its solution is given by

$$\int M(\text{treat y constant})dx + \int N(\text{terms free from x})dy = c$$

$$\therefore \quad \int [y\cos x + \sin y + y]\,dx + \int o\,dy = c$$

$$\therefore \quad \text{y} \cdot \int y\cos x \cdot dx + \sin y \cdot \int dx + y \cdot \int dx = c$$

$$\therefore \quad y\sin x + x\sin y + xy = c$$

Which is the require general solution

Check Your Progress:

Solve:(1) $\left(a^2 - 2xy - y^2\right)dx - (x+y)^2 \cdot dy = 0$

Ans. $a^2x - x^2y - xy^2 - \dfrac{y^3}{3} = c$

(2) $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$

Ans. $x + y \cdot e^{x/y} = c$

(3) $[\cos x \cdot \tan y + \cos(x+y)]dx + [\sin x \cdot \sec^2 y + \cos(x+y)]dy = o$

Ans. $\sin x \cdot \tan y + \sin(x+y) = c$

(4) $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy \cdot e^{xy^2} - 3y^2\right)dy = 0$

Ans. $e^{xy^2} + x^4 - y^3 = c$

(5) $[1 + \log(xy)]dx + \left\{1 + \dfrac{x}{y}\right\}dy = 0$

Ans. $y + x\log(xy) = c$

(6) $\left(2xy + e^y\right)dx + \left(x^2 + xe^y\right) \cdot dy = 0$

Ans. $x^2y + xe^y = c$

(7) $[y\sin(xy) + xy^2\cos(xy)]dx + [x\sin(xy) + x^2y\cos(xy)]dy = 0$

Ans. $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = \sin xy + xy\cos(xy) + 2xy\cos(xy) - x^2y^2\sin(xy)$

General solution is given by

$$xy\sin\ (xy) = c$$

7.5 LET US SUM UP

In this chapter we have learn

- ❖ solution of D.E:- general solution, particular solution
- ❖ variable separable form:- dx

$\varsigma f(x)dx = \varsigma f(y)dy + c$

- ❖ Equations reducible to variable separable form.
- ❖ Homogeneous differential equation i.e $\frac{dy}{dx} = \frac{f(xy)}{g(xy)}$

With substituting Y=Yx.

7.6 UNIT END EXERCISE

Solve the following differential equation.

i. $\frac{dy}{dx} = \frac{sinx + xcosx}{Y(1+2logu)}$

ii. $\frac{dy}{dx} + x^2 = x^2 e^3 y$

iii. $2x\,Cosy\,dx - (1+x^2)siny\,dy = 0$

iv. $(x+1)\frac{dy}{dx} + 1 = e^{-2y}$

v. $\frac{dy}{dx} = ax + by + c$

vi. $\frac{dy}{dx} = sin(x+y) + cos(x+y)$

vii. $\frac{dy}{dx} = e^{y/x} + y/x$

viii. $\frac{dy}{dx} = (4x + y + 1)^2$

ix. $\frac{dy}{dx} = y/x + sin\left(y/x\right)$

x. $\frac{dy}{dx} = (x + y + 1)^2$

xi. $\frac{dy}{dx} = 1 + y/x - Cos\,y/x$

xii. $(x^3 + y^3)\frac{dy}{dx} = x^2 y$

xiii. $\left(4 - \frac{y^2}{x^2}\right)dx + \frac{2y}{x}dy = 0$

xiv. $$\frac{dy}{dx} + \frac{x^2 + 3y^2}{3x^2 + y^2} = 0$$

xv. $$4(x+y)\frac{dy}{dx} = 3x - 4y$$

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8

EQUATION REDUCIBLE TO EXACT EQUATIONS

UNIT STRUCTURE

8.1OBJECTIVE

After going through this chapter you will able to

- ❖ Find the solution of non-exact .differential equation.
- ❖ Find the solution of linear .differential equation.
- ❖ Reducing to non-linear equation into linear equation.
- ❖ Find the solution of non-linear equation.

8.2 INTRODUCTION

In previous chapter we have learn about exact differential equation & its solution. Now here we are going to discuss none exact differential equation. To find the solution of non-exact differential equation we use integrating factor which convert non-exact differential equation to exact differential equation. Also we discuss about solution of linear differential equation.

In some cases equations which are not exact can be converted to exact differential equation by multiplying by some suitable factor called as Integrating factor.

8.3 DEFINITION

Integrating Factor

If the equation leMdx +leNdy=0 is exact

then le is said to be an integrating factor of the equation Mdx+ Ndy = 0

8.3.1 Rules of finding Integrating factor :-

Rule (1)

If the equation Mdx+Ndy=0 is homogeneous then $\frac{1}{Mx+Ny}$ is integrating factor

Solved Example:

Example 1: $\left(x^2y-2xy^2\right)$ dx-$\left(\mathrm{x}^3-3x^2y\right)dy=0$

Solution: The given equation is

$\left(x^2y-2xy^2\right)$ dx-$\left(\mathrm{x}^3-3x^2y\right)dy=0$........(1)

This is a homogeneous equation.

Comparing with Mdx +Ndy=0 ; we have

$$M=x^2y-2xy^2$$

$$N=-\left(\mathrm{x}^3-3x^2y\right)$$

$$\therefore \text{ I.f. } =\frac{1}{Mx+Ny}$$

$$=\frac{1}{\mathrm{x}^3y-2x^2y-x^3y+3x^2y^2}$$

$$\therefore \frac{\left(\mathrm{x}^2y-2xy^2\right)}{x^2y^2}\text{ dx-}\frac{\left(\mathrm{x}^3-3x^2y\right)}{x^2y^2}\text{ dy}=0\text{ is exact}$$

$$\text{i.e. }\left(\frac{1}{\mathrm{y}}-\frac{2}{x}\right)\text{ dx-}\left(\frac{\mathrm{x}}{\mathrm{y}^2}-\frac{3}{y}\right)\text{ dy}=0\text{ is exact}$$

Its general solution is given by

$$\int M\ \left(\text{treat y const}\right)\text{ dx }+\int \mathrm{N}\ \left(\textit{terms free from x}\right)\text{ dy}=\mathrm{c}$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}\,dy=c$$

$$\therefore\ \frac{1}{y}\cdot\int dx-2\int\frac{1}{x}\,dx+3\cdot\int\frac{1}{y}\,dy=c$$

$$\therefore\ \frac{x}{2}-2\log x+3\log y=c$$

This is required general solution

Check your progress:

Solve

i) $\left(3xy^2-y^3\right)dx+\left(xy^2-2x^2y\right)dy=0$

Hint : I.F. $=\dfrac{1}{x^2y^2}$

General solution is given by

$$\frac{cy^2}{x^3}=c^{y/x}$$

ii) $\left(x^2-3xy+2y^2\right)dx+x\left(3x-2y\right)dy=0$

Hint: I.F. $=\dfrac{1}{x^3}$

General solution is given by

$x^2\log x+3xy=y^2+cx^2$

8.3.2 Rule (II) :

If the equation Mdx+Ndy=0 can be written as

$M=y\,f_1(xy)dx,\quad N=xf_2(xy)\cdot dy=0$

i.e. $M=y\,f_1(xy),\quad N=x\,f_2(xy)$

then $\dfrac{1}{Mx-Ny}$ is an integration factor.

Note :- $f_1(x\,y),\ f_2(x\,y)$ are functions of xy.

Solved Examples :-

Example 2: Solve $\left(x^2y^2+2\right)ydx+\left(2-2x^2y^2\right)xdy=0$

Solution**:** The equation is given by

$$\left(x^2y^2+2\right)ydx+\left(2-2x^2y^2\right)xdy=0$$

Comparing with Mdx+Ndy=0; we have

$\therefore\ M=\left(x^2y^2+2\right)y$

$N=\ \left(2\text{-}2x^2y^2\right)\cdot x$

$$I.f.=\frac{1}{Mx-Ny}$$

$$\therefore\ \text{I.f. } =\frac{1}{xy\left(x^2y^2+2-2+2x^2y^2\right)}$$

$$I.f.=\ \frac{1}{3x^3y^3}$$

$$\therefore\quad \frac{\left(x^2y^2+2\right)y}{3x^3y^3}dx+\frac{\left(2\text{-}2x^2y^2\right)\cdot x}{3x^3y^3}dy=0$$

$$i.e\ \left(\frac{1}{3x}+\frac{2}{3}\cdot\frac{1}{x^3y^2}\right)\text{dx } + \ \left(\frac{2}{3x^3y^3}-\frac{2}{3y}\right)\ \cdot\text{dy}=0$$

which is a exact equation

$\therefore$ Its General solution is given by

$$\int M\ \left(\textit{treat}\ \text{y constant}\right)\text{dx }+\ \int N\ \text{(terms free from x) dy}=c$$

$$\therefore\ \int\left(\frac{1}{3x}+\frac{2}{3}\cdot\ \frac{1}{x^3y^2}\right)\text{dx }+\ \int\text{-}\frac{2}{3y}\cdot\ \text{dy}=c$$

$$\therefore\quad \frac{1}{3}\int\frac{1}{x}dx+\frac{2}{3y^2}\cdot\ \int\frac{1}{x^3}dx-\frac{2}{3}\int\frac{1}{y}\cdot dy=c$$

$$\therefore\frac{1}{3}\log x-\frac{2}{6x^2y^2}-\frac{2}{3}\log y\ \ =c$$

$$\therefore\ \log x\text{-}\frac{1}{x^2y^2}-2\log y=c_1\ \text{ where } c_1=\ 2c$$

Check your progress:

1. solve :

$$\left(x^2y^2+xy+1\right)\ y\cdot dx+\left(x^2+y^2-xy+1\right)\ xdy=0$$

Hint: I.F. $\dfrac{1}{2x^2y^2}$

G.S. is given by

$$xy\ +\log x\text{-}\frac{1}{xy}-\log y=c$$

2. $\qquad y\left(xy+2x^2y^2\right)+x\ \left(xy\text{-}x^2y^2\right)\ dy=0$

Ans $\qquad x^2=cy\cdot e^{\frac{1}{xy}}$

8.3.3 Rule (III):

If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=a$ function of x alone. Say f(x) then $e\int f(x)\,dx$ is integrated. factor.

Solved Examples :-

Example 3: Solve $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$

Solution: The given equation is

$$(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$$

Comparing with Mdx+Ndy=0; we get

$$M=\ y^4+2y$$

$$N=\ xy^3+2y^4-4x$$

$$\therefore\ \frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\ y^4+2y\right)$$

$$\frac{\partial M}{\partial y}=4y^3+2$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\ xy^3+2y^4-4x\right)$$

$$\frac{\partial N}{\partial x}=y^3-4$$

$$\therefore\ \frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$$

$$=\ \frac{-3\cdot\left(y^3+2\right)}{y\left(y^3+2\right)}$$

$$=-\frac{3}{y}=\text{ function of y alone}$$

$$\therefore\ \text{I.F.}\ =\ e^{\int f(y)\cdot dy}$$

$$=\ e^{-3\cdot\int\frac{1}{y}dy}$$

$$=\ e^{-3\log y}$$

$= e^{\log\left(\frac{1}{y^3}\right)}$

$I.F. = \dfrac{1}{y^3}$

$\therefore \dfrac{\left(y^4 + 2y\right)}{y^3} dx + \dfrac{\left(xy^3 + 2y^4 - 4x\right)}{y^3} \cdot dy = 0$

This is exact differential equation

Comparing with Mdx+ Ndy=0; we get

$$M = y + \frac{2}{y^2}$$

$$N = x + 2y - 4\frac{x}{y^3}$$

General solution is given by

$$\int M \left(treat \text{ y constant}\right) \text{dx} + \int N \left(\text{terms free from x}\right) \text{dy} = \text{c}$$

$$\therefore \int\left(\text{y} + \frac{2}{\text{y}^2}\right)\text{dx} + 2 \cdot \int \text{y dy} = \text{c}$$

$$\therefore \left(\text{y} + \frac{2}{\text{y}^2}\right)\int \text{dx} + 2\frac{\text{y}^2}{2} = c$$

$$\left(\text{y} + \frac{2}{\text{y}^2}\right)\text{x} + \text{y}^2 = c$$

This is required general solution.

Check your progress:

Solve :

i) $\left(2xy^4e^y + 2xy^3 + y\right)\text{ dx} + \left(x^2y^4e^y - x^2y^2 - 3x\right)\text{ dy} = 0$

Hint : I.F. $= \dfrac{1}{y^4}$..........

General solution is given by

$x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

ii) $x^2y^3dx + \left(x^3y - 2\right)\text{ dy} = 0$

Ans $3x^3y - 2y - 6 = cy \cdot e^{3/y}$

8.4 LINEAR EQUATION AND EQUATIONS REDUCIBLE TO LINEAR FORM

The first order and first degree linear -
Differential equation is of the type

$$\frac{d\,y}{d\,x}+py=Q$$

Where y is dependent variable and x is independent variable. and p& Q are functions of x only. (may be constant)

The above differential equation is known as Leibnitz's linear differential equation.

Working Rule:

1) Consider linear differential equation.

$$\frac{d\,y}{d\,x}+py=Q$$

Where P and Q are function of x or constants only

Its integrating factor is given by

$$I.F. = e^{\int p dx}$$

Its solution is given by

$$y\cdot\left(\text{I.F.}\right)=\int Q\cdot\left(I.F.\right)dx+c$$

Where c is arbitrary constant.

2) For linear differential equation

$$\frac{dx}{dy}+p_1x=Q_1$$

Where p_1 and Q_1 are functions of y or constants only

Its integrating factor is given by

$$\therefore\ \text{I.F.} = e^{\int p_1 dy}$$

Its solution is given by

$$x\cdot\left(IF\right)=\int Q\ (IF)\ \text{dy}+\text{c}$$

Where c is arbitrary constant.

Solved Examples:-

Example 4: Solve $\left(x+1\right)\ \frac{\text{dy}}{\text{dx}}-y=e^{x}\left(x+1\right)^{2}$

Solution: The given equation is

$$\left(x+1\right)\ \frac{\text{dy}}{\text{dx}}-y=e^{x}\left(x+1\right)^{2}$$

Dividing throughout by $\left(x+1\right)$ we have

$$\therefore\ \frac{\text{dy}}{\text{dx}}-\frac{1}{\left(x+1\right)}\cdot y=e^{x}\ \left(\text{x}+1\right).........(1)$$

This is of the type

$$\therefore\ \frac{\text{dy}}{\text{dx}}+py=Q$$

Hence equation (1) is linear differential equation.

Where

$$P=-\frac{1}{(x+1)}\ ,\ Q=\mathrm{e}^{x}\left(x+1\right)$$

$$\therefore\ \text{I.F.}\ =\mathrm{e}^{\int \mathrm{pdx}}$$

$$=e^{-\int\frac{1}{x+1}\,dx}$$

$$=\ \mathrm{e}^{-\log\,(x+1)}$$

$$I.F.\ =e^{\log\left(\frac{1}{x+1}\right)}$$

$$I.F.=\ \frac{1}{\mathrm{x}+1}$$

Hence the solution of differential equation (1) is

$$y\cdot\ (\text{I.F.})=\int Q\ (\text{IF})\ dx+c$$

$$\therefore\ \mathrm{y}\cdot\ \frac{1}{\mathrm{x}+1}=\int e^{x}\left(x+1\right)\ \frac{1}{(\mathrm{x}+1)}dx+c$$

$$\therefore\ \frac{\mathrm{y}}{\mathrm{x}+1}=\int e^{x}\cdot dx+c$$

$$\therefore\ \frac{\mathrm{y}}{\mathrm{x}+1}=\ \mathrm{e}^{x}+c$$

$$\therefore\ \mathrm{y}=\left(\mathrm{e}^{x}+c\right)\cdot\left(x+1\right)$$

This is the required solution.

Example 5: Solve $\left(1+y^{2}\right)\ \mathrm{dx}\ =\ \left(\tan\mathrm{y}^{-1}-x\right)\ \mathrm{dy}$

Solution: The given equation is

$$\therefore\ \ \left(1+y^{2}\right)\ \mathrm{dx}\ =\ \left(\tan\mathrm{y}\ \ -x\right)\ \mathrm{dy}$$

$$\therefore\ \ \frac{\mathrm{dx}}{\mathrm{dy}}=\frac{\tan y^{-1}-x}{1+y^{2}}$$

$$\therefore\ \ \frac{\mathrm{dx}}{\mathrm{dy}}=\frac{\tan^{-1}y}{1+y^{2}}-\frac{1}{1+y^{2}}\cdot\ \mathrm{x}$$

$$\therefore\ \ \frac{\mathrm{dx}}{\mathrm{dy}}+\frac{1}{1+y^{2}}\cdot\ \mathrm{x}=\frac{\tan^{-1}y}{1+y^{2}}\text{..........(1)}$$

This is of the type

$$\frac{dx}{dy}+px=Q$$

$$Where\ \mathrm{p}=\frac{1}{1+y^{2}},Q=\frac{\tan^{-1}y}{1+y^{2}}$$

Hence equation (i) is a linear differential equation

$$\therefore\ \text{I.f}\ =\mathrm{e}^{\int \mathrm{pdy}}$$

$$=e^{\int\frac{1}{1+y^{2}}\cdot dy}$$

$$I.F.\ =e^{\tan^{-1}y}$$

The solution of differential equation (i) is

$$x(I.F.) = \int Q\ (\text{I.F.})\,dy + c$$

$$\therefore\ \text{x} \cdot \text{e}\ \tan^{-1} y = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y} \cdot dy + c$$

consider the integral

$$\int \frac{\tan^{-1} y}{1+y^2}\ e^{\tan^{-1} y} \cdot dy$$

put $\text{z} = \tan^{-1} y$

Differentiating with respect to z

$$1 = \frac{1}{1+Y^2} \cdot \frac{dy}{dz}$$

$$\therefore\ \frac{1}{1+Y^2} \cdot dy = dz$$

$$\therefore\ \int \text{z} \cdot \text{e}^{\text{z}} \cdot dz$$

$$= z \cdot \int e^z \cdot dz - \left(\int \frac{d}{dz}\ \text{z} \int \text{e}^z \cdot dz \right) \text{dz}$$

$$=\ \text{z} \cdot \text{e}^{\text{z}} - \int l \cdot e^z \cdot dz$$

$$=\ \text{z} \cdot \text{e}^{\text{z}} - e^z$$

$$=\ \text{e}^{\text{z}} (z-1)$$

put $\text{z} = \tan^{-1} y$

$$=\ \text{e}^{\tan^{-1} y} \left(\tan^{-1} y - 1 \right)$$

$\therefore$ solution is given by

$$\text{x} \cdot \text{e}^{\tan^{-1} y} = e^{\tan^{-1} y} \left(\tan^{-1} y - 1 \right) + c$$

$$\therefore\ \text{x} = \tan^{-1} y - 1 + c \cdot e^{-\tan^{-1} y}$$

This is the required solution.

Example 6: Solve

$$x\left(1-x^2\right)\frac{dy}{dx} + \left(2x^2-1\right)y = x^3$$

Solution: The given equation is

$$x\left(1-x^2\right)\frac{dy}{dx} + \left(2x^2-1\right)y = x^3$$

$\div$ through out by $\text{x}(1\text{-x}^2)$ we have

$$\therefore\ \frac{\text{dy}}{\text{dx}} + \frac{\left(2x^2-1\right)}{x\left(1-x^2\right)} y = \frac{x^3}{x\ \left(1\text{-x}^2\right)} \text{..........(1)}$$

Hence equation (1) is linear in dependent variable y

This is of the type

$$\frac{dy}{dx}+py=Q$$

$$where\ \mathrm{P}=\frac{\left(2x^2-1\right)}{x\left(1-x^2\right)},Q=\frac{x^3}{x\left(1-x^2\right)}$$

$$\therefore\ \mathrm{I.F.}=\mathrm{e}^{\int \mathrm{pdx}}$$

$$Let\ \mathrm{P}=\frac{2x^2-1}{x\left(1-x\right)\left(1+x\right)}$$

$$P=-\frac{1}{x}+\frac{1}{2\left(1-x\right)}-\frac{1}{2\left(1+x\right)}$$

$(By\ partial$ fraction)

$$\therefore\ \mathrm{IF}\ =\ \mathrm{e}^{\int\left[-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right]\cdot dx}$$

$$=\ \mathrm{e}^{-\log x+\frac{1}{2}\log(1-x)-\frac{1}{2}\log(1+x)}$$

$$=\ \mathrm{e}^{-\left[\log x\cdot\sqrt{1-x^2}\right]}$$

$$=\ \mathrm{e}^{\log\cdot\left[x\cdot\sqrt{1-x^2}\right]^{-1}}$$

$$IF=\frac{1}{x\sqrt{1-x^2}}$$

Hence solution of differential equation (i) is

$$y\ \left(\mathrm{IF}\right)=\int Q\left(IF\right)dx+c$$

$$\therefore\ \mathrm{y}\cdot\frac{1}{\mathrm{x}\sqrt{1-\mathrm{x}^2}}=\int\frac{x^{\not{3}2}}{\left(1-x^2\right)}\cdot\frac{1}{\not{x}\sqrt{1-x^2}}\cdot dx+c$$

$$=\int\frac{x}{\left(1-x^2\right)^{3/2}}\cdot dx+c$$

$$=-\frac{1}{2}\cdot\int\left(-2x\right)\left(1-x^2\right)^{3/2}\cdot dx+c$$

$$=-\frac{1}{2}\left[\frac{\left(1-x^2\right)^{-1/2}}{-1/2}\right]+c$$

$$\left\{\int \mathrm{f}^{\mathrm{n}}\cdot f^1=\frac{f^{n+1}}{n+1}\right.$$

$$\therefore\frac{y}{x\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}+c$$

$$\therefore\ \mathrm{y}=\mathrm{x}+\mathrm{cx}\sqrt{1-x^2}$$

Which is the required solution.

Example 7: Solve

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy}=1$$

Solution: The given equation is

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy}=1$$

$$\therefore\ \frac{\mathrm{dy}}{\mathrm{dx}}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\text{........(1)}$$

Which is of the type

$$\frac{\mathrm{dy}}{\mathrm{dx}}+py=Q$$

$$\text{where P}=\frac{1}{\sqrt{\mathrm{x}}},Q\ =\ \frac{\mathrm{e}^{-2\sqrt{\mathrm{x}}}}{\sqrt{\mathrm{x}}}$$

The equation (1) is linear in y

$$\therefore\ \text{I.F.}\ =\mathrm{e}^{\int pdx}$$

$$=e^{\int\frac{1}{\sqrt{x}}\cdot dx}$$

$$I.F.=e^{2\sqrt{x}}$$

Hence the solution of differential equation (1) is

$$y\cdot\left(IF\right)=\int Q\ \cdot\left(\mathrm{IF}\right)dx+c$$

$$y\cdot e^{2\sqrt{x}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\cdot e^{2\sqrt{x}}\cdot dx+c$$

$$=\int\frac{1}{\sqrt{x}}dx+c$$

$$y\cdot e^{2\sqrt{x}}=2\sqrt{x}+c$$

This is the required general solution.

Example 8: Solve $\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\cdot\frac{dy}{dx}$

Solution: The given equation is

$$\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\cdot\frac{dy}{dx}$$

$$\therefore\ \left(\text{x-e}^{\tan^{-1}y}\right)\cdot\frac{dy}{dx}=-\left(1+y^2\right)$$

$$\therefore\ \text{x-e}^{\tan^{-1}y}=-\left(1+y^2\right)\cdot\frac{dx}{dy}$$

$$\therefore\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}\text{........(1)}$$

Which is of the type

$$\frac{\mathrm{dx}}{\mathrm{dy}}+px=Q$$

$where\ \mathrm{p} = \frac{1}{1+\mathrm{y}^2}, Q = \frac{e^{\tan^{-1} y}}{1+y^2}$

The equation (1) is linear differential equation

Hence

$$IF = \mathrm{e}^{\int \mathrm{pdy}}$$

$= \mathrm{e}^{\int \frac{1}{1+y^2} \cdot dy}$

$IF = e^{\tan^{-1} y}$

Hence solution of differential equation (1) is given by

$$x \cdot (IF) = \int Q\ \text{(IF) dy} + \text{c}$$

$$\mathrm{x} \cdot \mathrm{e}^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} \cdot dy + c$$

$$\mathrm{x} \cdot \mathrm{e}^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1+y^2} \cdot dy + c ....(2)$$

$put\ \tan^{-1} y = t$

$\therefore\ \frac{1}{1+\mathrm{y}^2} \cdot dy = dt$

$\therefore$ equation (2) becomes

$\mathrm{x} \cdot \mathrm{e}^{\tan^{-1} y} = \int e^{2t} \cdot dt + c$

$\mathrm{x} \cdot \mathrm{e}^{\tan^{-1} y} = \frac{e^{2t}}{2} + c$

$put\ \mathrm{t} = \tan^{-1} y$

$\therefore x \cdot e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + c$

$\therefore 2x \cdot e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c_1$ where $\mathrm{c}_1 = 2c$

This is the required general solution.

Check your progress:

1) Solve

i) $\left(2y + x^2\right) dx = xdy$

Ans: $\mathrm{y} = \mathrm{x}^2 \log(cx)$

ii) $\frac{dy}{dx} + \frac{y}{1-x} = x^2 - x$

Ans : $2\mathrm{y} = (1\text{-}\mathrm{x})\left(c^2 - x^2\right)$

iii) $\left(\mathrm{x}^2 + 1\right) \cdot \frac{\mathrm{dy}}{dx} = \mathrm{x}^3 \text{-} 2\mathrm{xy} + \mathrm{x}$

Ans : $\left(\mathrm{x}^2 + 1\right)\mathrm{y} = \frac{x^4}{4} + \frac{\mathrm{x}^2}{2} + c$

iv) $$\frac{dy}{dx}+\frac{x}{\left(1-x^2\right)^{3/2}}\cdot y=\frac{x\left(1+\sqrt{1-x^2}\right)}{\left(1-x^2\right)^2}$$

Hint I.F. $=e^{-\frac{1}{\sqrt{1-x^2}}}$

$$y=\frac{1}{\sqrt{1-x^2}}+c\cdot c^{\frac{1}{\sqrt{1-x^2}}}$$

v) $$dx+xdy=e^{-y}\sec^2\cdot dy$$

Hint : I.F. $=e^{y}$

$$x\cdot e^{y}=\tan y+c$$

vi $$x\cos x\cdot\frac{dy}{dx}+\left(\cos x-x\sin x\right)\cdot y=1$$

Hint : I.F. $=\frac{x}{\sec x}$

$$xy\cos x=x+c$$

vii $$\left(x^2+1\right)^3\cdot\frac{dy}{dx}+4x\cdot\left(x^2+1\right)^2\cdot y=1$$

Hint : If $=\left(x^2+1\right)^2$

$$\left(x^2+1\right)^2\cdot y=\tan^{-1}x+c$$

viii $$\left(x+y+1\right)\cdot\frac{dy}{dx}=1$$

Hint : I.F. $=e^{-y}$

$$x+y+2=c\cdot e^{y}$$

ix $$\left(x+2y^3\right)\cdot dy=ydx$$

Hint I.F. $=\frac{1}{y}$

$$x=y^3+cy$$

8.5 EQUATIONS REDUCIBLE TO LINEAR FORM

I) Bernoulli’s Equation :

The equation of the form

$$\frac{dy}{dx}+py=Q\cdot y^n$$

is called as Bernoulli's equations

$\div$ throughout by y^n, *we get*

$$\therefore\ y^{-n}\cdot\frac{dy}{dx}+P\cdot y^{1-n}=Q.........(1)$$

$Let\ y^{1-n} = u$

$$\therefore\ (1\text{-n}) \cdot y^{-n} \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$u \sin g$ equation (1) we get

$$\therefore\ \frac{1}{1\text{-n}} \cdot \frac{du}{dx} + Pu = Q$$

$$\therefore\ \frac{du}{dx} + (1\text{-n}) \cdot pu = (1-n)\ Q$$

This is Bernoulli's differential equation and can be solved.

Note: The equation is also Bernoulli's equation

We divide by x^n and substitute u= x^{1-n} and proceed.

Solved Examples:-

Example 9: Solve $\frac{dy}{dx} + \frac{y}{x} = xe^x \cdot y^2$.

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = xe^x \cdot y^2 \text{.............(1)}$$

Which is of the type

$$\frac{\text{dy}}{\text{dx}} + Py = Q\ \cdot y^n \text{........}$$

$Where\ \text{p} = \frac{1}{\text{x}},\ \text{Q}\ = \text{xe}^{\text{x}}, n = 2$

Equation (1) *is* Bernoulli's differential equation

$\div$ throughtout by y^2, *we get*

$$\therefore\ y^{-2} \cdot \frac{dy}{dx} + \frac{1}{x} \cdot y^{-1} = x \cdot e^x \text{.................(2)}$$

$Put\ \ y^{-1} = u$

Differentiating with respect to x

$$\therefore\ -1 \cdot y^{-2} \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore\ y^{-2} \cdot \frac{dy}{dx} = -\frac{du}{dx}$$

$u \sin g$ equation 2 we get

$$\therefore\ -\frac{\text{du}}{\text{dx}} + \frac{1}{x} \cdot u = x \cdot e^x$$

$$\therefore\ \frac{\text{du}}{\text{dx}} - \frac{1}{x} \cdot u = x \cdot e^x$$

Which is linear differential equation.

where $\text{p} = \frac{1}{\text{x}}, Q = \text{-x} \cdot \text{e}^{\text{x}}$

$$\therefore I.F. = e^{\int Pdx}$$

$= e^{-\int \frac{1}{x} dx}$

$= e^{-\log x}$

$I.F. = e^{\log\left(\frac{1}{x}\right)}$

$\therefore$ I.F. $= \frac{1}{x}$

Hence, General solution is given by

$$u \cdot (IF) = \int Q \ (\text{IF}) \ \text{dx} + \text{c}$$

$$\therefore \ u \cdot \frac{1}{x} = \int \text{-x} \ e^{x} \ \frac{1}{x} \ \text{dx} + \text{c}$$

Put $u = y^{-1}$

$\therefore \ y^{-1} \cdot \frac{1}{x} = -\int e^{x} dx + c$

$\frac{1}{xy} = -e^{x} + c$

This is the required solution.

Example 10: Solve $xy\left(1 + xy^{2}\right) \cdot \frac{dy}{dx} = 1$

Solution: The given equation is

$$xy \cdot \left(1 + xy^{2}\right) \cdot \frac{dy}{dx} = 1$$

$$\therefore \frac{dx}{dy} = xy + x^{2}y^{3}$$

$$\therefore \frac{dx}{dy} - xy = x^{2}y^{3} ....(1)$$

which is of the type,

$$\frac{\text{dx}}{\text{dy}} + px = Q \ \cdot \text{x}^{\text{n}}$$

where $\text{p} = \text{-y}, \ Q \ = \text{y}^{3}, n = 2$

Equation 1 is a Bernoulli's differential equation

$\div$ through out by x^{2}, *we get*

$\therefore \ x^{-2} \cdot \frac{dx}{dy} - x^{-1} \cdot y = y^{3} ....(2)$

Let $x^{-1} = u$

$$\therefore \ -x^{-2} \cdot \frac{dx}{1y} = \frac{du}{dy}$$

$\therefore \ x^{2} \cdot \frac{\text{dx}}{\text{dy}} = -\frac{du}{dy}$

equation (2) *becomes*

$$-\frac{du}{dy}-uy=y^3$$

$$\therefore\ \frac{du}{dy}+uy=-y^3$$

which is a linear differential equation.

$\text{p}=\text{y},\ \ \text{Q}\ =-y^3$

$\therefore\ \text{I.F.}=\text{e}^{\int pdy}$

$=\ \text{e}^{\int y\cdot dy}$

$I.F.=\ \text{e}^{y^2/2}$

Hence general solution is given by

$$u\cdot(IF)=\int Q\ \cdot(IF)\cdot dy+c$$

$$u\cdot e^{y^2/2}\ =\int -y^3\cdot\ e^{y^2/2}\ \cdot dy+c$$

$$Let\ \frac{\text{y}^2}{2}=t$$

$$\therefore y\ dy=dt$$

$$\therefore\ \text{u}\cdot e^{y^2/2}\ =-\int 2t\cdot\ e^t\cdot dt+c$$

$$\text{u}\cdot e^{y^2/2}\ =-2\left[t\cdot e-e^t\right]+c$$

$$Put\ \text{u}=\text{x}^{-1},\ \text{t}=y^2/2$$

$$\therefore\ \ \frac{1}{\text{x}}\cdot e^{y^2/2}\ =-2\left[\frac{y^2}{2}\cdot e^{y^2/2}\ -e^{y^2/2}\ \right]+c$$

$$\therefore\ \ \frac{1}{\text{x}}\cdot e^{y^2/2}\ =-y^2\cdot e^{y^2/2}\ +2\cdot e^{y^2/2}\ +c$$

$$\therefore\ \ \frac{1}{\text{x}}\cdot e^{y^2/2}\ +y^2\cdot e^{y^2/2}\ -2\cdot e^{y^2/2}\ =c$$

This is the required general solution.

Check your progress:

i) solve:-

i) $$\frac{dy}{dx}-y\tan x=y^4\sec x$$

$\text{Hint: If }=\sec^3 x$

$$\frac{\sec^3 x}{y^3}+3\tan x+\tan^3 x=c$$

ii) $$\frac{dy}{dx}-xy=y^2\cdot e^{-x^2/2}\cdot\log x$$

$\text{Hint: I.F. }=e^{x^2/2}$

$$\frac{1}{y}\cdot e^{-x^2/2}+x\log x-x=c$$

iii) $xy-\frac{dy}{dx}=y^3\cdot e^{-x^2}$

Hint: If $= e^{x^2}$

$e^{x^2}=y^2\left(2x+c\right)$

iv) $x\frac{dy}{dx}+3y=x^4 e^{1/x^2}\cdot y^3$

ans $y^2+x^6\left(e^{1/x^2}+c\right)=1$

v) $2xdx-y^2\left(y^3+x^2\right)\cdot dy=0$

Hint : If $=e^{-y^3/3}$

$x^2=c\cdot e^{y^3/3}-y^3-3$

vi) $\frac{dy}{dx}=e^{x-y}\left(e^x-e^y\right)$

Hint: IF $=e^{e^x}$

$e^y=c\cdot e^{-e^x}+e^x-1$

$\frac{dy}{dx}=2y\left(1-2xy\right)$

Hint :-I.F. $=e^{2x}$

$\frac{1}{y}=\left(2x-1\right)+c\cdot e^{-2x}$

8.5.1 (II) Equation of the type :

The equation $f^1\left(x\right)\cdot\frac{dy}{dx}+p\cdot f\left(y\right)=Q$

Where P and Q are functions of x can be reduced to linear by substituting f(y) =u and equation becomes

$$\frac{du}{dx}+pu=Q$$

Similarly the equation

$$f^1\left(x\right)\cdot\frac{dy}{dx}+p\,f(x)=Q$$

Can be reduced to linear by substituting f (x)=u

Solved Examples:-

Example 12: Solve $\sin y\cdot\frac{dy}{dx}=\left(1-x\cos y\right)\cdot\cos y$

Solution:

The given equation is

$$\sin y\cdot\frac{dy}{dx}=\left(1-x\cos y\right)\cdot\cos y$$

$$\therefore \ \sin y \cdot \frac{dy}{dx} = \cos y - x\cos^2 y$$

$\div$ throughout by $\cos^2 y$, *we* get

$$\therefore \ \frac{\sin y}{\cos^2 y} \cdot \frac{dy}{dx} = \frac{\cos y}{\cos^2 y} - x$$

$$\therefore \ \sec y \cdot \tan y \cdot \frac{dy}{dx} - \sec y = -x ....(1)$$

which is of the form

$$f^1(y) \cdot \frac{dy}{dx} + pf(y) = Q$$

where f(y) = secy, p = -1. Q = -x

Let secy = u

Differentiating with respect to x

$$\therefore \ \sec y \cdot \tan y \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$\therefore$ *equation* (1) becomes

$$\therefore \ \frac{du}{dx} - u = -x$$

Which is a linear differential equation.

Where p = -1, Q = -x

$$\therefore \ I.F. = e^{\int p dx}$$

$$= \ e^{-x}$$

$$I.F = e^{-x}$$

Hence General solution is given by

$$u \cdot (IF) = \int Q \cdot (IF) \ dx + c$$

$$u \cdot e^{-x} = \int -x \cdot e^{-x} \cdot \ dx + c$$

$$= \ -\int x \cdot e^{-x} \cdot \ dx + c$$

$$u \cdot e^{-x} = -\left[x\left(-e^{-x}\right) - 1 \cdot e^{-x} \right] + c$$

Put u = sec y

$$\therefore \ \sec y = x + 1 + c \cdot e^x$$

This is required general solution

Check Your Progress:

1) Solve :

I) $\dfrac{dy}{dx} + \dfrac{1}{x}\tan y = \dfrac{1}{x^2}\tan y \cdot \sin y$

Hint $\div$ throughout by $\tan y \cdot \sin y$

$$\frac{1}{x\sin y} = \frac{1}{2x^2} + c$$

ii) $\frac{dy}{dx} - x^3 \cos^2 y = -x \sin 2y$

H int $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

$\div$ through out by $\cos^2 y$

$IF = e^{x^2}$

$2\tan y = x^2 - 1 + c_1 \cdot e^{-x^2}$

8.6 LET US SUM UP

In this chapter we have learned

- ❖ Integrating factor for non-exact equation.
- ❖ Using integrating factor find the solution of non-exact equation.
- ❖ Using integrating factor find the solution of linear differential equation.
- ❖ Bernoulli's equation.

8.7 UNIT END EXERCISE

Solve the following D.E:

i. $\frac{dy}{dx} + \frac{4x}{(x^2+1)} y = \frac{1}{(x^2+1)^3}$

ii. $\frac{dy}{dx} + x^2 y = x^5$

iii. $\frac{dy}{dx} + \frac{(1-2x)}{x^2} y = 1$

iv. $(1+y^2)dx = (\tan^{-1} y - x)dy$

v. $(x^2 + y^2 + 1)dx - 2xy\, dy = 0$

vi. $(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$

vii. $(x^2 + y^2)dx - (x^2 + xy)dy = 0$

viii. $y(1+xy)dx + (1-xy) \times dy = 0$

ix. $(2y^2 + 4x^2 y)dx + (4xy + 3x^3)dy = 0$

x. $\frac{dy}{dx} + (\cot x)y = \cos x$

xi. $\frac{dy}{dx} + y \sec x = \tan x$

xii. $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

xiii. $(1+x^2)\frac{dy}{dx} + y = e^{\tan -1} x$

xiv. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

xv. $\sec x\, dy = (y + \sin x)dx$

xvi. $(y \log x - 1)y\, dx = x\, dy$

xvii. $\frac{dy}{dx} + xy = x^3 y^3$

xviii. $\frac{dy}{dx} + \frac{xy}{1-x^2} = xy^{1/2}$

xix. $y - Cosx\,\frac{dy}{dx} = y^2(1 - Sinx)Cosx$

xx. $y\,dx + x(1 - 3x^2 y^2)dy = 0$

*****

9

APPLICATIONS OF DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

9.1 OBJECTIVE

After going through this chapter you will able to

❖ Use differential equation to find the equation of any curve.

❖ Use differential equation physics like projectile motion, S.H.M's, Rectilinear motion, Newton's law of cooling.

❖ Use differential equation in electric circuits.

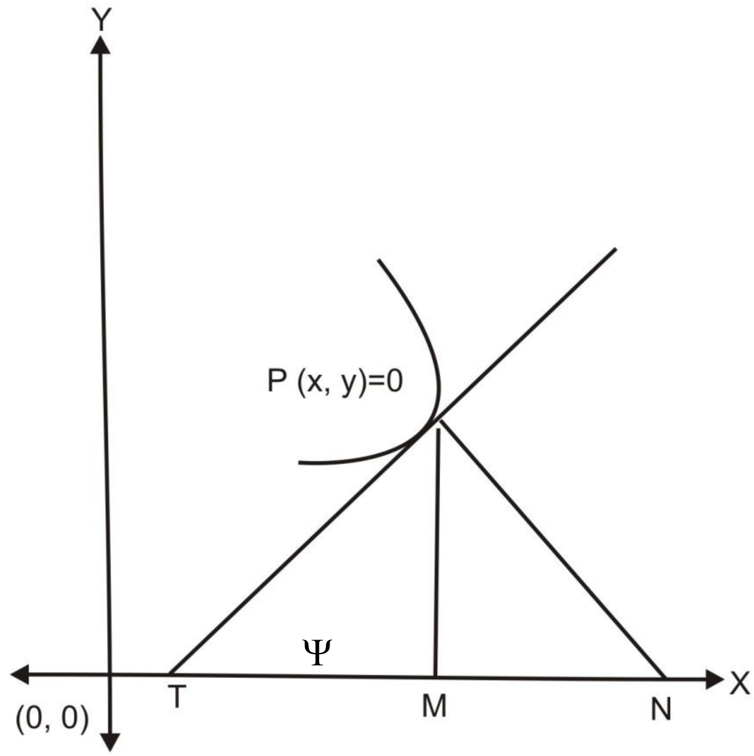
9.2 INTRODUCTION

In previous chapter we have learn to solve differential equations. We differ type. Now here we are going use differential equation in different field its useful to geometrical, physical, and electronic circuits, civil engineering and so on we are going to discuss few application of differential equation.

9.3 GEOMETRICAL APPLICATIONS

Cartesian Co-ordinates:

Let f (x_1 y_1) = 0 be the equation of the curve Let p (x_1 y_1) i.e. any point on it.



The tangent and normal at p meet X axis in T and respectively.

Let PM Ψ X axis

Let ∠ MTP = Ψ

∴ ∠ MTP = Ψ ...... [Geometrical Construction]

Then,

Slope of Tangent at p = tan Ψ = $\left(\frac{dy}{dx}\right)(x_1, y_1)$

Equation of tangent at p is $y - y_1 = \left(\frac{dy}{dx}\right)(x - x_1)$

X intercept of tangent = $x_1 - y_1\left(\frac{dx}{dy}\right)p$

$$= x_1 - \frac{y_1}{\left(\frac{dy}{dx}\right)_p}$$

y □intercept of tangent = $y_1 - x_1\left(\frac{dy}{dx}\right)P$

Equation of the normal at P is given by

$$y - y_1 = -\left(\frac{dx}{dy}\right)(x - x_1)$$

6) Length of tangent = PT = $y_1\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

7) Length of Normal at $P = PN = y_1\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

8) Length of Sub tangent $= \frac{y_1}{\left(\frac{dy}{dx}\right)}$

9) Length of Sub normal $= y_1 \cdot \left(\frac{dy}{dx}\right)$

10) If e is a radius of curvature at p then

$$e = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Solved Examples:

Example 1:

Find the curve which passes through the points [2, 1] and [8, 2] for which sub tangent at any point varies as the abscissa of that point.

Solution: Let p (x,y) be a point on the curve

We know,

$$\text{subtangent} = \frac{y}{\frac{dy}{dx}}$$

From given condition-

$$\frac{y}{\frac{dy}{dx}} \propto x$$

$$\therefore \frac{y}{\frac{dy}{dx}} = kx.....[k = \text{constant}]$$

$$\therefore y = kx\frac{dy}{dx}$$

$$\frac{1}{x}dx = \frac{k}{y}dy$$

$$k\frac{dy}{y} = \frac{1}{x}dx$$

which is in variable seperate form

integrate both side

$$\therefore k.\int \frac{1}{y}\, dy + \text{constant}$$

k. log y =log x + log c

$$\therefore \log y^{k} = \log(cx)$$

$$\therefore y^{k} = (cx).......(1)$$

The Curve passes through the points [2,1] and [6, 2]

put x = 2 , y = 1, in equation [1]

$$\therefore 1^{k} = 2c$$

$$\therefore 1 = 2c$$

$$\therefore c = \frac{1}{2}$$

put x = 8, y = 2, in eq^{n} [1]

$$\therefore 2^{k} = c \text{ x } 8$$

$$2^{k} = \frac{1}{2} \text{ x } 8^{4}$$

$$2^{k} = 4$$

$$\therefore 2^{k} = 2^{2}$$

$$\therefore K = 2$$

put Value of C and K in eq^{n} [1]

$\therefore\ y^2 = \frac{1}{2}x$

$\therefore\ 2y^2 = x$

This is the equation of the Curve

Example 2: Find the curves in which the length of the radius of curvature at any point is equal to two times the length of the normal at that point.

Solution: Let p [x , y] be a point on the curve

We Known that,

$$\text{Radius of curvature} = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\text{Lenght of normal} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$\therefore$ From given condition -

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = 25\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} \bullet \left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 25\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 25 \bullet \frac{d^2y}{dx^2} \longrightarrow [\,1\,]$$

Let $\frac{dy}{dx} = z$

$$\therefore \frac{dy^2}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(z)$$

$$= \frac{dz}{dy} \bullet \frac{dy}{dx}$$

$$= \frac{dz}{dy} \bullet z$$

$$\therefore \frac{d^2y}{dx^2} = z\frac{dz}{dy}$$

From eq^n (1)

$$\therefore 1 + z^2 = 25 \cdot z \cdot \frac{dz}{dy}$$

$$\therefore 2zy \cdot \frac{dz}{dy} = 1 + z^2$$

$$\therefore \frac{2z}{1 + z^2} \cdot dz = \frac{1}{y} \cdot dy$$

which is in variable separable form

$\therefore$ Integrate both side

$$\therefore \int \frac{2z}{1 + z^2} \, dz = \int \frac{1}{y} \cdot dy \text{ constant}$$

$$\therefore \log\left(1+z^2\right) = \log y + \log c$$

$$\therefore \log\left(1+z^2\right) = \log (cy)$$

$$1+z^2 = cy$$

$$\therefore z^2 = cy - 1$$

$$\therefore z = \sqrt{cy - 1}$$

Again put $z = \frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \sqrt{cy - 1}$$

$$\therefore \frac{1}{\sqrt{cy - 1}} \, dy = dx$$

This is in variable separable form

$\therefore$ Integrate both sides

$$\therefore \frac{1}{c}\int \frac{c \cdot 1}{\sqrt{cy - 1}} \cdot dy = \int dx + \text{constant}$$

$$\therefore \frac{1}{c} \cdot 2\sqrt{cy - 1} = x + c_1$$

$\therefore\ 2\sqrt{cy-1} = cx + cc_1$

$\therefore\ 2\sqrt{cy-1} = cx + c_2$

Where $c_2 = cc_1$

which is the eq^n of the curve.

Check your progress:

1) Determine the curves for which sub normal is the arithmetic mean between the abscissa and the ordinate

[Hint :

$$y\,\frac{dy}{dz} = \frac{x}{z} + \frac{y}{z}\ ;\ \text{simplify}$$

Equation is homegeneous.

Ans : $(x+2y)\cdot(x+y)^2 = c$

9.4 PHYSICAL APPLICATION

Rectilinear Motion:

It is a Motion of a body of Mass in start moving from a fixed point O along a straight line OX under the action of a force F. Let p be the position of the body at any instant

Where OP = X , then

1) velocity $v = \frac{dx}{dt}$

2) The acceleration $= \frac{dv}{dt}$

$$= \frac{d^2x}{dt^2}$$

$$= v\cdot\frac{dv}{dx}$$

By chain rule -

$$\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$$

$$= \frac{dv}{dx}\cdot v$$

$$= v\cdot\frac{dv}{dx}$$

3) Newton's second law of motion is given by

$$f = ma$$
$$= m \cdot \frac{dv}{dt}$$
$$= m \cdot \frac{d^2x}{dt^2}$$
$$f = mv \cdot \frac{dv}{dx}$$

where f = effective force

D' Alembert's principle:-

Algebraic sum of the forces acting on a body along the given direction is equal to the product of mass and acceleration in that direction.

$$\text{ie } m \cdot \frac{d^2x}{dt^2} = \sum F$$

$$\therefore \sum F - m\frac{d^2x}{dt^2} = O$$

Solved examples:

Example 3:

A moving body is opposed by a force per unit mass of a value CX and resistant per unit mass value bv, where X and V are the displacement and velocity of the particle at that instant. Show that the velocity of the particle. If it starts from rest, is given by.

$$v^2 = \frac{c}{2b^2}\left(1 - e^{2bx}\right) - \frac{cx}{b}$$

Solution: Consider the motion

Step 1) :

Let m be the mass of the particle moving to right. Now the opposing forces mcx and mbv^2 will act to the left.

$$mcx \longleftarrow mv \cdot \frac{dv}{dx} \longrightarrow$$

$$mbv^2 \longleftarrow$$

ie □mcx and □mbv^2 are forces to the right

By D' Alembert's principle

$$mv \cdot \frac{dv}{dx} = -mcx - mbv^2$$

step[2]

$$\therefore v\frac{dv}{dx} + bv^2 = -cx \longrightarrow [1]$$

Let $v^2 = z$

$$\therefore 2v \cdot \frac{dv}{dx} = \frac{dz}{dx}$$

$\therefore eq^n$ [1] becomes

$$\frac{1}{2} \cdot \frac{dz}{dx} + bz = -cx$$

$$\therefore \frac{dz}{dx} + 2bz = -2cx$$

which is a linear equation in z

$\therefore$ p = 2b. Q = -2cx.

$$\therefore \text{I.F.} = e^{\int pdx}$$

$$= e^{\int 2bdx}$$

$$\text{I.F.} = e^{2bx}$$

Its general solution is given by

$$z\,[\,IF\,] = \int Q \cdot (IF)\, dx + \text{constant}$$

$$\therefore z e^{2bx} = \int (-2cx) e^{2bx} \cdot dx + c_1$$

$$= -2c \cdot \int x \cdot e^{2bx} \cdot dx + c_1$$

$$= -2c \cdot \left[x \int e^{2bx} \cdot \left(dx - \int \frac{d}{dn} x \int e^{2bx} \cdot dx \right) \right] + c_1$$

$$= -2c \cdot \left[x \cdot \frac{e^{2bx}}{2b} \int 1 \cdot \frac{e^{2bx}}{2b} dx \right] + c_1$$

$$z e^{2bx} = -2c \left[\frac{x \cdot e^{2bx}}{2b} - \frac{1}{2b} \cdot \frac{e^{2bx}}{2b} \right] + c_1$$

$$v^2 \cdot e^{2bx} = \frac{-cx \cdot e^{2bx}}{b} + \frac{c}{2b^2} \cdot e^{2bx} + c_1$$

$$\therefore v^2 = -\frac{cx}{b} + \frac{c}{2b^2} + c_1 \cdot e^{-2bx} \longrightarrow [3]$$

[III] to find c_1, we impose initial conditions

ie for x = o , v = o in eq^n [3]

$$o = 0 + \frac{c}{2b^2} + c_1$$

$$\therefore\ g_1 = -\frac{c}{2b^2}$$

put values of g_1 in eq^n [3]

$$v^2 = -\frac{cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2}\cdot e^{-2bx}$$

$$\therefore\ v^2 = -\frac{c}{2b^2}\left(1-e^{-2bx}\right) - \frac{cx}{b}$$

Example 4: A body of mass m. Falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity [ie kv^2]. If it falls through a distance x and possesses a velocity v at that instant show that

$$\frac{2k\,x}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right), \text{ where } mg = ka^2$$

Solution:

Step :1

Let the body of mass m fall from ‘O’

The forces acting on the body are

1) Its weight mg acting vertically downwards.

2) The resistance kv^2 of the air acting vertically upwards.

The net forces acting on the body vertically downwards

= mg □ kv^2 . . . . [mg □ ka^2 given]

= ka^2 □ kv^2

= k [a^2 □ v^2] . . . . . .[1]

Step [2] By D’Alembert’s Principle

$$mv\cdot\frac{dv}{dx} = k\left(a^2 - v^2\right)$$

$$\therefore\ \frac{v}{a^2 - v^2}\cdot dv = \frac{k}{m}\cdot dx$$

This is in variable separable form

Integrating both sides

$$\therefore\ -\frac{1}{2}\cdot\int\frac{-2v}{a^2-v^2}\cdot dv = \frac{k}{m}\cdot\int dx + c_1\ \ldots\ \left[c_1 = \text{ constant}\right]$$

$$\therefore\ -\frac{1}{2}\ \log\ \left(a^2 - v^2\right) = \frac{k}{m}x + c_1 \longrightarrow (2)$$

Step [3] To Final c_1 , we put initial conditions

ie when x = o , v = o.

$$\therefore \text{ From } (2)$$

$$\therefore -\frac{1}{2} \log a^2 = c_1$$

put value of c_1 in eq^n [2]

$$\therefore -\frac{1}{2} \log \left(a^2 - v^2\right) = \frac{k}{m} x - \frac{1}{2} \log a^2$$

$$\therefore -\frac{1}{2} \log \left(a^2 - v^2\right) = \frac{k}{m} x - \frac{1}{2} \log a^2$$

$$- \log \left(a^2 - v^2\right) = \frac{2kx}{m} - \log a^2$$

$$\therefore \log a^2 - \log \left(a^2 - v^2\right) = \frac{2kx}{m}$$

$$\therefore \frac{2kx}{m} = \log \left(\frac{a^2}{a^2 - v^2}\right)$$

Check your progress:

1) A particle of Unit mass is projected upward with velocity u and the resistance of air produces a □ retardation kv^2 and v is the velocity at any instant show that the velocity v with which the particle will return to the point of projection is given by

$$\frac{1}{v^2} = \frac{1}{u^2} + \frac{k}{g}$$

2) Determine the least velocity with which a particle must be projected vertically upwards so that it does not return to the Earth. Assume that it is acted upon by the gravitational attraction of the earth only.

Ans : Least Velocity $vo = \sqrt{2gR}$

R = Radius of earth

3) A paratrooper and his parachute weigh 50 kg. At the instant parachute opens. He is Travelling vertically downward at the speed of 20 m/s. If the Air resistance varies directly as the instantaneous velocity and its 20 Newtons. When the velocity is 10 m/s Find the limiting velocity, the position and the velocity of the paratrooper at any time “t”.

$$v = 5 \left[s - e^{-gt/25} \right] \ldots = 25 \text{ m/s}$$

$$x = 5\left[st + \frac{25}{g} \cdot e^{-gt/25} \right] + c_1$$

$$x = 25t - \frac{125}{g} \left[1 - e^{-gt/25} \right]$$

9.5 SIMPLE ELECTRIC CIRCUITS

The following Notations are frequently used. Units are given in Brackets .

t (seconds) $\longrightarrow$ *Time*

q (coulombs) $\longrightarrow$ *Charge on capacitor*

i (ampere) $\longrightarrow$ *Current*

e (volts) $\longrightarrow$ *voltage*

R (ohms) $\longrightarrow$ *Resistance*

L (Hentries) $\longrightarrow$ *Induatance*

C (Farads) $\longrightarrow$ *capacitance*

$\therefore$ Current is the rate of electricity

$\therefore$ $i = \frac{dq}{dt}$

[II] Current at each point of a network is got from Kirchhoff's laws :

1) The algebraic sum of the currents into any point is zero.

2) Around any closed path the algebraic sum of the voltage drops in any specific direction is zero.

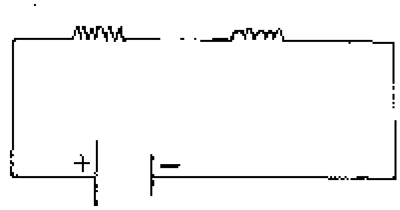
3) Voltage drops as current i flows through a resistance R is Ri ; through an induction L is $L\frac{di}{dt}$ and through a capacitor C is $\frac{q}{c}$.

Solved examples:

Example 5: A constant emf E volts is applied to a ckt. containing a constant resistance. R ohms in
series and a constant inductance L henries. It the initial current is zero, show that the current builds upto half its theoretical maximum
in $\frac{L \log 2}{R}$ seconds.

Solution:

Step (1) R



Let i be the current in the circuit at any time 't' .

The by Kirchoff's law, we have

$$E = L \cdot \frac{di}{dt} + Ri$$

$$\therefore\ L\frac{di}{dt} + Ri = E$$

$$\therefore\ \frac{di}{dt} + \frac{R}{L} \cdot i = \frac{E}{L} \longrightarrow (1)$$

Which is a linear equation in i .

$\therefore$ P = $\frac{R}{L}$ $\cdot$ $Q = \frac{E}{L}$

$\therefore$ I.F. = $e^{\int pdt}$

$= e^{\int \frac{R}{L} \cdot dt}$

I.F. = $e^{\frac{R}{L} \cdot t}$

$\therefore$ The general solution is given by

$$i\ .(IF) = \int Q\ .(IF) \cdot dt + \text{constant}$$

$$\text{i. } e^{\frac{R}{L} \cdot t} = \int \frac{E}{L} \cdot e^{\frac{R}{L} \cdot t} \cdot dt + c$$

$$= \frac{E}{L} \cdot \frac{L}{R} \cdot e^{\frac{R}{L} \cdot t} + c$$

$$i.\ e^{\frac{R}{L} \cdot t} = \frac{E}{R} \cdot e^{\frac{R}{L} \cdot t} + c$$

$$\therefore\ i = \frac{E}{R} + c \cdot e^{\frac{-R}{L} \cdot t} \longrightarrow (2)$$

To find c , we impose initial $\cdot$ conditions

i.e. at t = o , i = o

$\therefore$ O = $\frac{E}{R} + C$

$\therefore$ C = $-\frac{E}{R}$

$\therefore$ *Equation* (2) becomes

i = $\frac{E}{R} - \frac{E}{R} \cdot e^{-\frac{R}{L} \cdot t}$

$\therefore$ i = $\frac{E}{R}\left(1 - e^{-\frac{R}{L} \cdot t}\right) \longrightarrow (3)$

This is the expression for i at any time t.

Now as t increases decreases $e^{-\frac{R}{L} \cdot t}$ increases and its maximum value is $\frac{E}{R}$

Step (2)

Let the current in the circuit be half its theoretical maximum after a time T seconds then.

From eqn (3) $$\frac{1}{2}\ \frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}\cdot t}\right)$$

$$\therefore\ \frac{1}{2} = 1 - e^{-\frac{R}{L}\cdot t}$$

$$\therefore\ e^{-\frac{R}{L}\cdot t} = 1 - \frac{1}{2}$$

$$\therefore\ -\frac{R}{L}\cdot t = \log\frac{1}{2}$$

$$= \log\frac{1}{2}$$

$$= \log 1 - \log 2$$

$$= O - \log 2$$

$$\not{-}\frac{R}{L}\cdot t = \not{-}\log 2$$

$$\therefore\ t = \frac{L\cdot\log 2}{R}$$

Check Your Progress:

1) The equation of the eml in terms of current i for an electrical circuit having resistance R and a condenser of capacity C, in series is.

$$E = Ri + \int\frac{i}{c}\cdot dt$$

Find the current i at any time t , when

E = Eo sin wt

Ans : $$i = \frac{WcEo}{\sqrt{1+R^2C^2W^2}}\cos\ (wt - \phi) + c_1\cdot e^{\frac{-t}{RC}}$$

where $\phi = \tan^{-1}$ (RCW)

2) An electrical circuit contains an inductance of 5 henries and on resistance of 120 in series with an emf 120 sin (20t) Volts. Find current if it is zero when

t = o ; at t = 0.01

Ans : $$\frac{20}{10144}\cdot\left[12\sin(0.2) - 100\cos\cdot(0.2) + 100.e\frac{-3}{125}\right]$$

9.6 NEWTON'S LAW OF COOLING

The law states that the rate at which the temp of a body changes is proportional to the difference between the instantaneous temp of the body and the temp of the surrounding medium.

If Q is the instantaneous temp of the body an Qo the temp of the surrounding then.

$$\frac{dQ}{dt}\alpha\left(Q-Qo\right)$$
$$\frac{dQ}{dt}=-k\left(Q-Qo\right)$$

Where k is a constant and Q decreases as t increases i.e. $\frac{dQ}{dt}$ is negative hence negative sign is added.

Solved Examples:

Example 6: The temperature of the air is 30^0C.and the substance cools from 100^0C to 70^0C in 15 minutes, find when the temperature will be 40^0C.

Solution:

$$\textit{Initially at } \mathrm{t}=0,\ \mathrm{T}=100$$
$$\therefore\ \log\left(100-20\right)=0+c_1$$
$$\therefore c_1=\log\ 80$$
$$\textit{Put value } \text{of } \mathrm{c}_1 \text{ in eq}^{\mathrm{n}}\ (1)$$
$$\therefore\ \log\left(\mathrm{T}-20\right)\ =-\ kt+\log\ 80$$
$$\therefore\ k\mathrm{t}\ =\log 80-\log\ \left(\mathrm{T}-20\right)\longrightarrow\ (2)$$
$$\textit{When } \mathrm{t}=1,\quad \mathrm{T}=60^0\ c$$

$$\therefore\ \text{from } (2)$$
$$\mathrm{k}=\log\ 80\ -\ \log 40\longrightarrow(3)$$
$$\text{Divide eq}^{\mathrm{n}}\ (2) \text{ by } (3) \text{ we have}$$
$$t=\frac{\log 80-\log(T-20)}{\log 80-\log 40}\longrightarrow(4)$$
$$\textit{when } \mathrm{t}=2\min\textit{ute},T=?$$
$$\textit{from}\ \text{eq}^{\mathrm{n}}\ (4) \text{ we have}$$
$$2=\frac{\log\left(\frac{80}{T-20}\right)}{\log\left(\frac{280}{40}\right)}$$
$$4=\frac{80}{\mathrm{T}-20}$$
$$4=\frac{80}{\mathrm{T}-20}$$

$$4T-80=80$$
$$4T=160$$
$$T=\frac{160^{40}}{4^{1}}$$
$$T=\ 40^0\ \mathrm{c}$$

$\therefore$ *The temp of* the body at the end of second minute will be 40^0c

Check your progress:

(i) A body at temperature 100^0 c is placed in a room whose temp is 20^0c and cools to 60^0 c in 5 minutes. Find its temp. after a fruther interval of 3 minutes.

Ans :- 46.4^0 c.

9.7 LET US SUM UP

In this chapter we have learn Application of Differential equation like-

- ❖ Geometrical Application:- like to find the equation of curve the equation of normal.
- ❖ Physical application: 1) Rectilinear motion.
  2) D'Alembert's Principle.
- ❖ In electronics Circuits.
- ❖ Newton's law of cooling.

9.8 UNIT END EXERCISE

i. Find the equation of the curve whose slope is equal to $\frac{y+3}{x+2}$ at every point of it and which passes through the point (0,0).

ii. A curve passing though (3, 0) has as gradient $7/2$ at this point such that at every point on it $\frac{d^2y}{dx^2} = x$. Find the equation of the curve.

iii. If the slope of the curve at any point is $\frac{y^2 logx - y}{x}$ and the curve passes through the point (1,1). Find its equation.

iv. The curve i in an electric circuit containing resistance R and self-inductance satisfies the differential equation $L\frac{di}{dt} + Ri = E\ Sin.wt$ where R, E & W are constant. If i=0 find the current at time t.

v. The change θ of a condenser, capacity C, discharged in a circuit of resistance .R and self-inductance .L satisfies the differential equation

$L\frac{d^2\theta}{dt^2} + R\frac{d2}{dt} + \frac{\theta}{c} = 0$ Solve the equation with initial conditions that $\theta = 0$, $\frac{d2}{dt} = 0$ when $t = 0_1$ and $CR^2 < 4L$.

vi. A radioactive substance decomposes at the rate proportional to the amount present at the time. How much mass will be left if initially a substance 2mg is supplied.

vii. The Newton's Law of Cooling states that the rate of cooling of a substance is proportional to the difference in the temperature of the body and that of the surrounding is 20. If water cools down to 60° in first 20minutes, during what time will it cool to 30° ?

viii. If $L\frac{di}{dt} = 30\ Sin\ 10\Pi t$ find i in terms of t given that L=2 and i=0, at t=0.

*****

10

SUCCESSIVE DIFFERENTIATION

UNIT STRUCTURE

10.1 OBJECTIVE

After going through this unit, you will be able to

- Find higher order derivative
- Formula of n^{th} order derivative
- Leibnitz's theorem
- Application of Leibnitz's theorem

10.2 INTRODUCTION

In this chapter we shall study the methods of finding higher ordered derivatives for a given functional expression.

This is done in two stages:

Stage I : We shall establish some standard results and solve some problems using these results.

Stage II : We shall prove Leibnitz theorem and using it find higher order derivatives of given function

Notation:- Different notations used for derivatives of y=f(x) with respect to x are

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, ... \frac{d^ny}{dx^n}, ...$ (Due to Lebinitz)

$y, y, y,$ (Due to Newton)

$\mathrm{f}(\mathrm{x}), f(x), \ldots\ldots f^{n}(x)$ (Due to langravge)

For convenience we also use notations

$\mathrm{y}_{1,} y_{2,} \ldots\ldots y_{n} \ldots\ldots or\ y^{'}, y^{''}, y^{'''} \ldots etc.$

$(y_{n})_{0} = value$ of n^{th} derivative of y at x=o

Stage (I)

10.3 SOME STANDARD RESULTS

(1) Let

$$y = e^{ax}$$

$$y_{1} = ae^{ax} \quad \mathrm{y}_{2} = a^{2} e^{ax} \ldots\ldots y_{n} = a^{n} e^{ax}$$

(2) Let

$$y = a^{mx}$$

$$y_{1} = ma^{mx} \ (\mathrm{loga}),\ \mathrm{y}_{2} = m^{2} a^{mx} (\log a^{2}) \ldots\ldots.$$

$$\mathrm{y}_{\mathrm{n}} = m^{n} a^{mx} (\log a)^{n}$$

3) $y = \sin(ax+b)$

$$y_{1} = a\cos\ (\mathrm{ax+b}) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

$$y_{2} = -a^{2}\sin\ (\mathrm{ax+b}) = a^{2}\ \sin\ \left[2\frac{\pi}{2} + (ax+b)\right] \ldots\ldots.$$

$$y_{n} = a^{n}\sin\ \left[(ax+b) + \frac{n\pi}{2}\right]$$

If a=l then

$y = \sin(x+b)$ and $y_{n} = \sin\left[(x+b) + \frac{n\pi}{2}\right]$

Also if b=0 then y= sin x and $y_{n} = \sin\left[x + \frac{n\pi}{2}\right]$.

4)If $y = \cos(ax+b)$ then on similar lines (m>n)

$$y_{n} = a^{n}\cos\left(ax + b + \frac{n\pi}{2}\right)$$

$y \;= (ax+b)^m$ (m>n)

$y_1 \;= ma(ax+b)^{m-1}$ (m is integer)

$y_2 \;= m(m-1)a^2 \; (ax+b)^{m-2}$

:

$y_n \;= m(m-1)(\text{m-2})......(\text{m-n+1}) \; \text{a}^{\text{n}} \; (ax+b)^{m-n}$

$y_n \;= n(n-1)(\text{n-2})......1.\text{a}^{\text{n}} \, (ax+b)^0 = n!a^n$

If m=n then

If a=1, b=0, then $y=x^n$

$y_n \;= n!$

6) $y \; = (ax+b)^{\text{-m}}$ (m is positive integer)

$y_1 \;= (-m)a(ax+b)^{-m-1}$

$y_2 \;= (-m)(-m-1)a^2 \; (ax+b)^{-m-2}$............

$y_n \;= (-m)(-m-1)......(\text{-m-n+1})\text{a}^{\text{n}} \; (ax+b)^{-m-n}$

$= (-1)m(\text{m+1})......(\text{m+n-1})\text{a}^{\text{n}} \; (ax+b)^{-m-n}$

7) $y = \dfrac{1}{ax+b}$

$y_1 = \dfrac{-a}{(ax+b)^2} = -a(ax+b)^{-2}$

$y_2 = (-a)a(-2)(ax+b)^{-3} = \dfrac{(-1)^2 a^2 2!}{(ax+b)^3}$.......

: : :

$y_n = \dfrac{(-1)^n n!a^n}{(ax+b)^{n+1}}$ if a=1 then y= $\dfrac{1}{x+b^3}$.....

$y_n = \dfrac{(-1)^n n}{(x+b)^{n+1}}$

8) $y = \log(ax+b)$

$y_1 = \dfrac{a}{(ax+b)}$

$$y_2 = \frac{-a^2}{(ax+b)^2} = \frac{(-1)^1 a^2}{(ax+b)^2} = \frac{(-1)^{2-1} a^2}{(ax+b)^2}$$

$$y_n = \frac{(-1)^{n-1} a^n}{(ax+b)^n}$$

9) $y = e^{ax} \cdot \sin(bx+c)$

$$y_1 = e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$$

Let $a = r\cos\alpha, b = r\sin\alpha,$

And $\therefore$ $r=\sqrt{a^2+b^2}$ $\alpha=\tan^{-1}\left(\frac{b}{a}\right)$

$$y_1 = e^{ax}\left[r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)\right]$$

$$= re^{ax}\left[\sin(bx+c+\alpha)\right]$$

$$y_1 = (a^2+b^2)^{1/2}\ e^{ax}\sin\left[bx+c+\tan^{-1}\left(\frac{b}{a}\right)\right]$$

similarly it can be proved that

$$y_2 = (a^2+b^2)^{2/2}\ e^{ax}\sin\left[bx+c+\tan^{-1}\left(\frac{b}{a}\right)\right]........$$

$$\therefore \quad y_n = (a^2+b^2)^{n/2}\ e^{ax}\sin\left[bx+c+\tan^{-1}\left(\frac{b}{a}\right)\right]$$

If a=1,b=1, c=0

$$y = e^{ax}\sin x$$

$$y_n = 2^{n/2}\ e^x\sin\left[x+\frac{n\pi}{4}\right]$$

$$y = e^{ax}\cos(bx+c)$$

$$y_n = (a^2+b^2)^{n/2}\ e^{ax}\cos\left((bx+c)+n\tan^{-1}\frac{b}{a}\right)$$

Type I

Example1: Find nth derivatives of the following :

i) $\sin^3 x$

ii) $\cos x \cos 2x \cos 3x$

iii) $y = e^x \cos x \cos 2x$

iv) $y = e^{x\cos\alpha}\cos(x\sin\alpha)$

Solution:

i) Let $y = \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ sin3x=3sinx-4sin$^3 x$

$$4\sin^3 x = 3\sin x - \sin 3x$$

$$\sin^3 x = \frac{1}{4}\cdot[3\sin x - \sin 3x]$$

Using the result for nth derivative of y=sin (ax+b) and nothing that nth derivative of sum or difference is sum or difference of nth derivatives, we get

$$y_n = \frac{1}{4}\left[3\sin\left(x + n\frac{\pi}{2}\right) - 3^n \sin\left(3x + \frac{n\pi}{2}\right)\right]$$

ii) Let

$$y = \cos x \cos 2x \cos 3x = \frac{1}{2}\cos 2x[\cos 4x + \cos 2x]$$

$$= \frac{1}{2}\cdot\cos 2x\cdot\cos 4x + \frac{1}{2}\cdot\cos 2x\cdot\cos 2x \quad \left[\because\ c_A c_B = \frac{1}{2}(c_{A+B} + c_{A-B})\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot[\cos 6x + \cos 2x] + \frac{1}{2}\cdot\frac{1}{2}[\cos 4x + \cos 0]$$

$$= \frac{1}{4}[\cos x + \cos 2x] + \frac{1}{4} + [1 + \cos 4x] = \frac{1}{4}[\cos 6x + \cos 2x] + \frac{1}{4}[1 + \cos 4x]$$

$$= \frac{1}{4}[\cos 6x + \cos 4x + \cos 2x + 1]$$

$$y_n = \frac{1}{4}\left[6^n \cos\left(6x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$$

iii) $y = e^x \cos x \cos 2x = \frac{e^x}{2}[\cos 3x + \cos x]$

$$= \frac{1}{2}\left[e^x \cos 3x + e^x \cos x\right]$$

[Using the result for nth derivative of y=e^{ax} cos (bx+c)

$$y_n = \frac{1}{2}\left[(10)^{n/2}\cdot e^x \cos\left(3x + \tan^{-1} 3\right) + (2)^{n/2}\cdot e^x \cos\left(x + \frac{n\pi}{4}\right)\right]$$ iv)

$$y = e^{x\cos\alpha}\cos(x\sin\alpha)$$

$[$*Here note* a=cosα, b=sinα, c=0$]$

$$\therefore\ y_n = \left(\cos^2\alpha + \sin^2\alpha\right)^{n/2} e^{x\cos\alpha}\cos\left[x\sin\alpha + n\tan^{-1}(\tan\alpha)\right]$$

$$\therefore\ y_n = e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha)$$

Example2

$If\ y = \sin px + \cos pxt, thanshow\ that : \mathrm{y_n} = p^n \left[1 + (-1)^n \sin 2px\right]^{1/2}$

Solution: $\therefore$ $\mathrm{y} = \sin px + \cos px$

$$\therefore \quad \mathrm{y_n} = p^n \sin\left(px + \frac{n\pi}{2}\right) + p^n \cos\left(px + \frac{n\pi}{2}\right) \qquad (\text{Results:3,4})$$

$$= p^n \left[\sin\left(px + \frac{n\pi}{2}\right) + \cos\left(px + \frac{n\pi}{2}\right)\right]$$

$$= p^n \left\{\left[\sin\left(px + \frac{n\pi}{2}\right) + \cos\left(px + \frac{n\pi}{2}\right)\right]^2\right\}^{1/2}$$

$$= p^n \left[1 + 2\sin\left(px + \frac{n\pi}{2}\right) \cdot \cos\left(px + \frac{n\pi}{2}\right)\right]^{1/2}$$

look at simplification: $(\text{a+b}) = \left[a + b) = \left[a + b)^2\right]^{1/2}\right]$

$$= p^n \left[1 + \sin(2px + n\pi)\right]^{1/2} \qquad \left[\because 2\mathrm{S_A} C_A = S_{2A}\right]$$

$$= p^n \left[1 + (-1)^n \sin 2px\right]^{1/2} \qquad \left[\because \mathrm{S_{A+B}} = S_A C_B + C_A S_B\right]$$

$\sin n\pi = 0$

and

$\cos nx = (-1)^n$

Example 3: If $y = (x-1)^n\ than$ show that $y + y_1 + \frac{y_2}{2!} + \frac{y_3}{3!} + ... + \frac{y_n}{n!} = x^n$.

Solution:

$$\therefore \quad y = (x-1)^n$$

$$\therefore \quad y_n = (x-1)^{n-1}$$

$$y_1 = n(x-1)^{n-1}$$

$$y_2 = n(n-1)(x-1)^{n-2}$$

$$\vdots$$

$$y_n = n\ !$$

$$\therefore y + y_1 + \frac{y_2}{2!} + \frac{y_3}{3!} + ... \frac{y_n}{n!}$$

$$= (x-1)^n + (x-1)^{n-1}(1) + \frac{n(n-1)}{2!}(x-1)^{n-2} + ... \frac{n\ !}{\mathrm{n}\ !}$$

$$= (\text{x-1})^n + n(x-1)^{n-1}(1) + \frac{n(n-1)}{2\ !}(x-1)^{n-2}(1)^2 + ...(1)^n$$

$$= \left[(\text{x-1}) + 1\right]^n$$

$$= (x)^n$$

$\left[(a+b)^n = a^n + na^{n-1}b \right.$ (Note: See Binomial expansion)

$+\frac{n(n-1)}{2!}a^{n-2}b^2 + ...b^n$

Hence $\left. a = x-1,, b=1 \right]$

Type II

Find nth derivatives by method of fraction :

Example 4: Find nth derivative of $y = \frac{x}{(x-1)(x-2)(x-3)}$.

Solution : Using method of partial fraction

$$y = \frac{x}{(x-1)(x-2)(x-3)}$$

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\therefore x = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

$Put\ x=1$ Put $x=2$ Put $x=3$

$$\therefore A = \frac{1}{2} \quad B = -2 \quad C = \frac{3}{2}$$

$$\therefore y = \frac{1}{2}\ \frac{1}{(x-1)} - \frac{2}{(x-2)} + \frac{3}{2}\ \frac{1}{(x-3)}$$

$$\therefore y_n = \frac{1}{2}\frac{(-1)^n\ n!}{(x-1)^{n+1}} - 2\frac{(-1)^n \cdot n!}{(x-2)^{n+1}} + \frac{3}{2}\cdot\frac{n!}{(x-3)^{n+1}}$$

10.4 TYPE III : USING COMPLEX NUMBERS

Example 5: Find nth derivative of $y = \frac{1}{x^2+a^2}$

Solution.:

We have $y = \frac{1}{(x-ai)(x+ai)}$

By partial fractions,

$$y=\frac{1}{2ai\left(x-ia\right)}-\frac{1}{2ai\left(x+ia\right)}$$

Using result (7) $\quad y_n=\frac{1}{2ai}\frac{\left(-1\right)^n n!}{\left(x-ia\right)^{n+1}}-\frac{1}{2ai}\frac{1}{\left(x+ia\right)^{n+1}}$

$$=\frac{\left(-1\right)^n\cdot n!}{2ia}\left[\frac{1}{\left(x-ia\right)^{n+1}}-\frac{1}{\left(x+ia\right)^{n+1}}\right]$$

To eliminate, we substitute

$$x+ia=-r\cdot e^{i\theta}\quad\therefore\ \text{x-ia}=\text{r}\cdot\text{e}^{-i\theta}$$

(1) Becomes,

$$=\frac{\left(-1\right)^n\cdot n!}{2ia}\left[\frac{i}{\left(re^{-\theta}\right)^{n+1}}-\frac{1}{\left(re^{+\theta}i\right)^{n+1}}\right]$$

$$=\frac{\left(-1\right)^n\cdot n!}{2ia\cdot r^{n+1}}\left[e^{i\left(n+1\right)\theta}-e^{-i\left(n+1\right)\theta}\right]$$

$$=\frac{\left(-1\right)^n\cdot n!}{ar^{n+1}}\left[\frac{e^{i\left(n+1\right)\theta}-e^{-i\left(n+1\right)\theta}}{2i}\right]\qquad \sin=\frac{e^{i\left(n+1\right)\theta}-e^{-i\left(n+1\right)\theta}}{2i}$$

$$\therefore\ y_n=\frac{\left(-1\right)^n\cdot n!}{ar^{n+1}}\cdot\sin\left[\left(n+1\right)\theta\right]$$

$$r=\sqrt{a^2+x^2},\theta=\tan^{-1\frac{a}{x}}$$

Example 6: Find nth derivative of $\quad y=\frac{x}{x^2+a^2}$

Solution.:

We have $\quad y=\frac{x}{\left(x-ia\right)\left(x+ia\right)}$

$$=\frac{ia}{\left(x-ia\right)\left(2ai\right)}+\frac{-ia}{\left(x+ia\right)\left(-2ai\right)}$$

$$\therefore y=\frac{1}{2}\left[\frac{1}{\left(x-ia\right)}+\frac{1}{\left(x+ia\right)}\right]$$

$$\therefore y_n=\frac{1}{2}\left[\frac{\left(-1\right)^n\cdot n!}{\left(x-ia\right)^{n+1}}+\frac{\left(-1\right)^n\cdot n!}{\left(x+ia\right)^{n+1}}\right]$$

Let $\quad \text{x+ia}=re^{i\theta},x-ia=re^{-i\theta}$

$\therefore\ \mathrm{r} = \sqrt{x^2 + a^2},\ \theta = \tan^{-1\left(\frac{4}{x}\right)}$

From $\quad \mathrm{y_n} = \dfrac{(-1)^n \cdot n!}{2}\left[\dfrac{1}{r^{n+1} \cdot e^{-i(n+1)\theta}} + \dfrac{1}{r^{n+1} \cdot e^{i(n+1)\theta}}\right]$

$$= \frac{(-1)^n \cdot n!}{r^{n+1}}\left[\frac{e^{i(n+1)\theta} + e^{-i(n+1)\theta}}{2}\right]$$

$$\therefore \mathrm{y_n} = \frac{(-1)^n \cdot n!}{r^{n+1}}\cos\left[(n+1)^{\theta}\right]$$

$\mathrm{r} = \sqrt{x^2 + a^2},\ \tan^{-1(a/x)}$

Example 7: Find nth derivative of tan^{-1} x

Solution.:

$$\mathrm{y} = \tan^{-1} x$$

$$y_1 = \frac{1}{x^2 + 1} = \frac{1}{(x-i)(x+i)}$$

$= \dfrac{1}{2i}\left[\dfrac{1}{(x-i)} - \dfrac{1}{(x+i)}\right] \quad \because \mathrm{is}(\text{n-1})^{th}$ *derivatives of* $\mathrm{y_1}$, we have

$$\mathrm{y_n} = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(X-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(X+i)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(X-i)^n} - \frac{1}{(X+i)^n}\right]$$

Let $x + 1 = re^{i\theta}, \quad \mathrm{x} - i = re^{-i\theta}$

$\therefore \quad r = \sqrt{1 + x^2}, \theta = \tan\left(\frac{1}{x}\right)$

$$\therefore \ \mathrm{y_n} = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{\left(re^{-i\theta}\right)^n} - \frac{1}{\left(re^{i\theta}\right)^n}\right]$$

$$y_n = \frac{(-1)^{n-1} \cdot (n-1)\ !}{r^n}\left[\frac{e^{in\theta} - e^{-in\theta}}{2\ !}\right]$$

where $\quad = \dfrac{(-1)^{n-1}(n-1)!}{r^n} \cdot \sin(n\theta)$

$r \quad = \sqrt{x^2 + 1} \qquad \theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

Example 8: Find n^th^ derivative of : $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

Solution. :

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$$

$$y=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$$

$$=\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

$$x=\tan\theta$$

$$y=\cos^{-1}(-\cos 2\theta)$$

$$=\cos^{-1}\left[\cos(\pi+2\theta)\right]$$

$$=\pi+2\theta$$

$=\pi+2\tan^{-1}x,$ from previous result.

$$y_n=2\frac{(-1)^{n-1}(n-1)!}{r^n}\cdot\sin n\alpha$$

Where $r=\sqrt{1+x^2}$ $\alpha=\tan^{-1}(1/x)$

Example 9: If y=sinx (sinx), Show that $\frac{d^2y}{dx^2}+\tan x\cdot\frac{dy}{dx}+y\cos^2 x=0$

Solution: $y=\sin(\sin x)$

$$\therefore \frac{dy}{dx}=\cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2}=-\sin(\sin x)\cos^2 x-\sin x\cdot\cos(\sin x)$$

$$\therefore \frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x$$

$$=-\sin(\sin x)\cos^2 x-\sin x\cdot\cos(\sin x)$$

$$+\tan x.\cos x.\cos(\sin x)+\cos^2 x\sin(\sin x)=0$$

Example **10:** If $p^2=a^2\cos^2\theta+b^2\sin^2\theta,$ Prove that $p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$

Solution.:

Diff. given relation twice,

$$2p\frac{dp}{d\theta}=-a^2 2\cos\theta\sin\theta+2b^2\sin\theta\cos\theta$$

$$\therefore \quad p\frac{dp}{d\theta}=\left(b^2\text{-a}^2\right)\sin\theta\,\cos\theta ---------------(1)$$

$$\frac{d^2p}{d\theta^2}+\left(\frac{dp}{d\theta}\right)^2=\left(b^2\text{-a}^2\right)\left(\cos^2\theta-\sin^2\theta\right)....................(2)$$

$$\therefore \quad p\frac{d^2p}{d\theta^2}+\frac{\left(b^2\text{-a}^2\right)\sin^2\theta\cos^2\theta}{p^2}=\left(b^2\text{-a}^2\right)\left(\cos^2\theta-\sin^2\theta\right)$$

$$\therefore \quad p^3\frac{d^2p}{d\theta^2}+p^2\left(b^2\text{-a}^2\right)\left(\cos^2\theta-\sin^2\theta\right)-\left(b^2\text{-a}^2\right)\sin^2\theta\cdot\cos^2\theta$$

$$=\left(b^2\text{-a}^2\right)\left[p^2\left(\cos^2\theta-\sin^2\theta\right)-\left(b^2\text{-a}^2\right)\sin^2\theta\cdot\cos^2\theta\right]$$

$$=\left(b^2\text{-a}^2\right)\left[\left(a^2\cos^2\theta+b^2\sin^2\theta\right)\left(\cos^2\theta-\sin^2\theta\right)-\left(b^2\text{-a}^2\right)\sin^2\theta\cdot\cos^2\theta\right]$$

$$=\left(b^2\text{-a}^2\right)\left[a^2\cos^4\theta-b^2\sin^4\theta\right]$$

$$=a^2b^2\left(\cos^4\theta+\sin^4\theta\right)-\left(a^4\cos^4\theta+b^4\sin^4\theta\right)$$

$$=a^2b^2\left[\left(\cos^2\theta+\sin^2\theta\right)^2-2\sin^2\theta\cdot\cos^2\theta\right]-\left(a^4\cos^4\theta+b^4\sin^4\theta\right)$$

$$=a^2b^2-\left(a^2\cos^2\theta+b^2\sin^2\theta\right)$$

$$=a^2b^2-p^4$$

$$\therefore \quad p^3\frac{d^2p}{d\theta^2}=a^2b^2-p^4$$

$$\therefore \quad p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$$

Example 11: If $y=\sin^{-1}\left(\frac{1+2\sin x}{2+\sin x}\right)$ then show that $\frac{dy}{dx}=\frac{\sqrt{3}}{2+\sin x}$

Solution.:

We have

$$\text{siny}=\frac{1+2\sin x}{2+\sin x}=\frac{2\sin x+4-3}{2+\sin x}=2-\frac{3}{2-\sin x}$$

$$\therefore \text{cosy}\frac{dy}{dx}=\frac{3\cos x}{(2+\sin x)^2}$$

Now $\text{cosy}=(1-\sin y)^{1/2}$

$$=\left[1\text{-}\frac{(1+2\text{sinx})^2}{(2+\text{sinx})^2}\right]^{1/2}$$

$$= \left[\frac{(2+\text{sinx})^2-(1+2\text{sinx})^2}{(2+\text{sinx})^2}\right]^{1/2}$$

$$= \frac{\sqrt{3}\cos x}{2+\sin x}$$

$$\frac{dy}{dx}=\frac{3\cos x}{(2+\sin x)^2}\cdot\frac{1}{\cos y}$$

$$=\frac{3\cos x}{(2+\sin x)^2}\cdot\frac{(2+\sin x)}{\sqrt{3}\cos x}$$

$$= \frac{\sqrt{3}}{2+\sin x}$$

Check Your Progress:

1) If y=xe^{y} then show that :

$$(1-y)\frac{d^2y}{dx^2}=(2-y)\left(\frac{dy}{dx}\right)^2$$

2) If $y=(1-x^2)^{1/2}\cdot\sin^{-1}x,$ then show that :3)

$$(1-x^2)\frac{d^2y}{dx^2}\times\frac{dy}{dx}+2x+y=0$$

$[H\text{int : sub x}=\sin\theta.\ \ \therefore \text{y=}\cos\theta$

$find\ \ \frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}........\ ]$

3) If $y=(\sin^{-1}x)^2$ *then show* that:

$$(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-2=0$$

$[H\text{int}:\frac{dy}{dx}=2\sin^{-1}x\cdot\frac{1}{\sqrt{1+x^2}}$

$\therefore(1-x^2)\left(\frac{dy}{dx}\right)^2=4y,\ \text{differentiate again}]$

4) If $y=xe^y,$ then show that:

$$(1-y)\frac{d^2y}{dx^2}=(2-y)\left(\frac{dy}{dx}\right)^2$$

$[H\text{int}:\ \text{Take log}]$

5) If $my=\sin(x+y)$

show that $\text{y}_2=-y(1+y_1)^3$, m is constant

6) Find n$^{\text{th}}$ derivative of $\sin^5 x\cdot\cos^3 x$

10.5 PROBLEMS

Type (I) :

In this type of problem we have to chosen one function as u and the other as v. If there is a polynomial in x then that is to be chosen as u and then apply Leibnitz theorem.

Example 12: Find n^th^ derivative of $x^2 e^x \cos x$

Solution:

Let $y = \left(x^2\right)\left(e^x \cos x\right)$

$u = x^2, v = e^x \cos x.$

Using standard result number

$$y_n = 2^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right).$$

Here $y = u \cdot v$

By Leibnitz theorem

$$y_n = uv_n + nc_1 u_1 v_{n-1} + nc_2 u_2 v_{n-2} + \ldots\ldots..$$

$$= x^2 \left[2^{n/2} e^x \cos x\left(x + \frac{rn}{4}\right)\right] + n\left(2x\right)\left[2^{\frac{n-1}{2}} e^x \cos\left(x + \frac{\left(n-1\right)}{4}\right)\right]$$

$$+ \frac{n(n-1)}{2\,!}\left(2\right)\cdot\left[2^{\frac{n-2}{2}} \cdot e^x \cos\left(x + \frac{\left(n-2\right)}{4}\pi\right)\right]$$

Example 12: If f(x)=tan x then show that

$$f^n\left(0\right) - {}^n c_2 f^{n-2}\left(0\right)^n + {}^n c_4 f^{n-4}\left(0\right)\ldots\ldots = \sin\left(\frac{n\pi}{2}\right)$$

Solution: $\cos x \cdot f\left(x\right) = \sin x$

Taking n^th^ derivatives both sides to the left side we apply Leibnitz theorem and to the right we use standard formula for n^th^ derivative of sinx, we get,

$$\cos x . f^n\left(x\right) +_n C_1 - \left(\sin x\right) f^{n-1}\left(x\right) + {}^n C_2\left(-\cos x\right) f^{n-2}\left(x\right) + \ldots\ldots = \sin\left(\frac{n\pi}{2}\right)$$

putting x=0 on both sides, we get,

$$f^n\left(0\right) + n_{C_2} f^{n-2}\left(x\right) + n_{C_4} f^{n-4}\left(0\right) + \ldots\ldots = \sin\left(\frac{n\pi}{2}\right)$$

Example 14: If $y = \dfrac{\log x}{x}$ then show that :

$$y_n = \frac{(-1)^n \cdot n!}{x^{n+1}}\left[\log x - \left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)\right]$$

Solution:

By standard results we have $u_n = \frac{(-1)^{n-1}(n-1)n}{x^n}$

$$v_n = \frac{(-1)^n \cdot n!}{x^{n+1}}$$

We have y=u. v

By Leibnitz theorem

$$y_n = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + ... + u_n v$$

$$= \log x \cdot \frac{(-1)^n .n!}{x^{n+1}} + n\frac{1}{x}\frac{(-1)^{n-1}(n-1)n}{x^n} + \frac{n(n-1)}{2!}\times$$

$$= \log x \cdot \frac{(-1)^n .n!}{x^{n+1}} + n\frac{1}{x}\frac{(-1)^{n-1}(n-1)n}{x^n} + \frac{n(n-1)}{2!}\times$$

$$\left[\because (-1)^{n-1} = (-1)^n \cdot (-1)^{-1} = -(-1)^n\right]$$

$$\therefore \; \mathrm{y_n} = \frac{(-1)^n .n!}{x^{n+1}}\left[\mathrm{logx}\text{-}\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{\mathrm{n}}\right)\right]$$

Example 15: If $I_n = \frac{d^n}{dx^n}\left(x^n \cdot \log x\right)$ then show that

(i) $I_n =_n 1_{n-1} + (n-1)!$ and

(ii) $\mathrm{I_n} = n!\left[\log x + 1 + \frac{1}{2} + ... + \frac{1}{n}\right]$

Solution:

(i) We have $\therefore \quad \mathrm{I_n} = \frac{d^n}{dx^n}\left(x^n \cdot \log x\right) = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}\left(x^n \log x\right)\right]$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[nx^{n-1}\log x + x^{n-1}\right]$$

$$= n\frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1}\cdot \log x\right] + \frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1}\right]$$

$$\therefore \quad \mathrm{I_n} \quad = n1_{n-1} + (n-1)!$$

$\left[\textit{if } \mathrm{n^{th}} \text{ derivative of } \mathrm{x^n} \textit{ is } \mathrm{n!} \text{ therefore } (\mathrm{n\text{-}1})^{th} \textit{ derivative of } x^{n-1} \text{is } (\mathrm{n\text{-}1})!\right]$

ii dividing (1) on both sides by n!

$$\frac{\mathrm{I_n}}{\mathrm{n!}} = \frac{nI_{n-1}}{n!} + \frac{(n-1)!}{n!}$$

i.e. $\frac{\mathrm{I_n}}{\mathrm{n!}} = \frac{I_{n-1}}{(n-1)!} + \frac{I}{n}$

Re *placing* n by $(\text{n-1})(n-2)\ldots\ldots\ldots..3,2,1$ *we get*

$$\frac{1_{(\text{n-1})}}{(\text{n-1})!} = \frac{1_{n-2}}{(n-2)!} + \frac{1}{(n-1)}$$

$$\frac{1_{(\text{n-2})}}{(\text{n-2})!} = \frac{1I_{n-3}}{(n-3)!} + \frac{1}{n-2}$$

$$\frac{1_2}{2!} = \frac{1_1}{1} + \frac{1}{2}$$

$$\frac{1_1}{1!} = \frac{1_0}{0!} + 1$$

Adding all the results columnwise we get,

$$\frac{1_{\text{n}}}{\text{n!}} + \frac{1_{n-1}}{(n-1)!} + \frac{1_{n-2}}{(n-2)!} + \ldots\ldots. + \frac{1_2}{2!} + \frac{1_1}{1!}$$

$$= \frac{1_{n-1}}{(n-1)!} + \frac{1_{n-2}}{(n-2)!} + \ldots\ldots. + \frac{1_1}{2!} + \frac{1_0}{0!}$$

$$+ \left[\frac{1}{\text{n}} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots\ldots. + \frac{1}{2} + 1\right]$$

cancelling common terms on both sides and noting that

$$\mathrm{I_0} = 0^{th} \text{ derivative of } \mathrm{x^0} \log \mathrm{x}$$

$$= \log x$$

and $0! = 1$ we get

$$\mathrm{I_n} = n! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots. + \frac{1}{\text{n}}\right]$$

Example 16: By forming in two different ways the $\mathrm{n^{th}}$ derivative of $\mathrm{x^{2n}}$ show that :

$$1 + \frac{\mathrm{n}^2}{1^2} + \frac{\mathrm{n}^2(n-1)^2}{1^2 - 2^2} + \frac{\mathrm{n}^2(n-1)^2(n-2)^2}{1^2, 2^2, 3^2} + \ldots . \frac{(2_n)!}{(n!)^2}$$

Solution:

Step 1: We have

$$y = x^{2n} \qquad \text{Standard formula}$$

$$\therefore \quad \mathrm{y_n} = (2n)(2n-1)\ldots\ldots..(2n-n+1)x^{2n-n}$$

$$= \frac{\left[(2n)(2n-1)\ldots\ldots..(n+1)\right]\left[n(n-1)\ldots\ldots 3,2,1\right]x^n}{\left[n(n-1)\ldots\ldots 3,2,1\right]}$$

$$=\frac{(2_n)!}{n!}x^n$$

Step 2: Again $\quad y=x^{2n}=\mathrm{x}^{\mathrm{n}}.\mathrm{x}^{\mathrm{n}}$

We apply Leibtnitz theorem to find y_n

$\mathrm{u}=\mathrm{x}^{\mathrm{n}},\mathrm{v}=\mathrm{x}^{n}$

$\mathrm{u}_{\mathrm{n}}=n!,\ \ \mathrm{v}_{\mathrm{n}}=n!$

$$\therefore\ \mathrm{y}_{\mathrm{n}}=\mathrm{x}^{\mathrm{n}}\cdot n!+{}^{n}C_1\left(nx^{n-1}\right)\left(n!x\right)+$$

$$\left(n(n-1)x^{n-2}\left(n!\frac{x^2}{2}\right)+..........+x^n.n!\right)$$ see note below

$$\therefore\ \mathrm{y}_{\mathrm{n}}=\mathrm{x}^{\mathrm{n}}.n!\left[\ 1+\frac{\mathrm{n}^2}{1^2}+\frac{\mathrm{n}^2(n-1)^2}{(2!)^2}+\frac{\mathrm{n}^2(n-1)^2(n-2)^2}{(3)^2}+....+\right]$$

$$=\mathrm{x}^{\mathrm{n}}.n!\left[\ 1+\frac{\mathrm{n}^2}{1^2}+\frac{\mathrm{n}^2(n-1)^2}{1^2.2^2}+\frac{\mathrm{n}^2(n-1)^2(n-2)^2}{1^2.\ 2^2.\ 3^2}+....+\right]$$

From (1) and (2)

$$\frac{(2n)\ !}{n!}x^n=\mathrm{x}^{\mathrm{n}}.n!\left[\ 1+\frac{\mathrm{n}^2}{1^2}+\frac{\mathrm{n}^2(n-1)^2}{1^2.2^2}+....\right]$$

$$=1+\frac{\mathrm{n}^2}{1^2}+\frac{\mathrm{n}^2(n-1)^2}{1^2.2^2}+\frac{\mathrm{n}^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+....=\frac{(2_n)!}{(n!)^2}$$

Note :

The n^{th} derivative of x^n is n! but $(n-1)^{th}$ derivative of x^n is not (n-1)! but n! .x.

To prove that this use the formula No.5 and put m = (n-1).

Type (II)

In this type of problems we (generally) proceed according to the following flow-diagram:

First express y in terms of x

↓

Differentiate both sides with respect to x and simplify

↓

Again Differentiate both sides and simplify

↓

Then apply Leibnitz theorem term by term and simplify to get the result.

Example 17: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ Show that

$$\left(x^2-1\right) y_{m+2} + \left(2n+1\right) xy_{n+1} + \left(n^2-m^2\right) y_n = 0$$

Solution:

$$y^{1/m} + y^{-1/m} = 2x$$

$$\therefore \quad y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$\left(y^{1/m}\right)^2 - 2xy^{\frac{1}{m}} + 1 = 0$$

is a quadratic equation in $y^{1/m}$

$$\therefore \quad y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\therefore \quad y^{1/m} = x + \sqrt{x^2-1} \qquad \left(\text{neglecting negative sign}\right)$$

$$\therefore \quad y = \left(x + \sqrt{\left(x^2-1\right)}\right)^{m}$$

$$\therefore \quad y_1 = m\left(x + \sqrt{\left(x^2-1\right)}\right)^{m-1} \cdot \left(1 + \frac{2x}{2\sqrt{\left(x^2-1\right)}}\right)$$

$$\therefore \quad y_1 = m\left(x + \sqrt{\left(x^2-1\right)}\right)^{m-1} \cdot \frac{\left(x + \sqrt{\left(x^2-1\right)}\right)}{\sqrt{\left(x^2-1\right)}}$$

$$\therefore \sqrt{x^2-1} - y_1 = m\left(x + \sqrt{\left(x^2-1\right)}\right)^{m}$$

$$= m \cdot y$$

$$\therefore \left(x^2-1\right) y^2{}_1 = m^2 y^2$$

Differentiating both the sides with respect to x, we get,

$$2\left(x^2-1\right) y_1 y_2 + 2xy^2{}_1 - 2m^2 yy_1 = 0$$

$$\therefore \ \left(x^2-1\right) y_2 + xy_1 - m^2 y = 0$$

Applying Leibnitz term by term to find n^th^ derivative we get,

$$\left[\left(x^2-1\right) y_{n+2} + n\left(2x\right) y_{n+1} + \frac{n\left(n-1\right)}{2!}\left(2\right) y_n\right] + \left[xy_{n+1} + n\left(1\right) y_n\right] - m^2 y_n = 0$$

$$\therefore \left(x^2-1\right) y_{n+2} + x\left(2n+1\right) y_{n+1} + \left(n^2-m^2\right) y_n = 0$$

Note : If we consider negative sign, we shall get the same result.

Example 18: If we $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^{n}$ *then* Show that

$$x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$$

Solution:

We have

$$\cos^{-1}\frac{y}{b}=n\log(\log x-\log n)^{n}$$

$$\therefore \qquad \mathrm{y}=\mathrm{b}\cos[n\log x-n\log n]$$

$$\therefore \qquad \mathrm{y}_1=-\mathrm{b}\sin[n\log x-n\log n]\left(\frac{n}{x}\right)$$

$$\therefore \qquad \mathrm{xy}_1=-n\mathrm{b}\sin[n\log x-n\log n]$$

differentiating both the sides with respect to x

$$\therefore \qquad \mathrm{xy}_2+\mathrm{y}_1=-n\mathrm{b}\cos[n\log x-n\log n]\cdot\left(\frac{n}{x}\right)$$

$$\therefore \qquad \mathrm{x}^2\mathrm{y}_2+xy_1=-n^2\left[\mathrm{b}\cos(n\log x-n\log n)\right]$$

$$\therefore \qquad =-n^2y$$

$$\mathrm{x}^2\mathrm{y}_2+xy_1+n^2y=0$$

Applying Leibnitz theorem term by term to differentiate n times, we get,

$$\left[x^2y_{n+2}+n2xy_{n+1}+\frac{n(n-1)}{2!}(2)y_n\right]+\left[xy_{n+1}+n(1)y_n\right]-n^2y_n=0$$

$$x^2y_{n+2}+x(2n+1)y_{n+1}+2n^2y_n=0$$

Example 19: If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then show that

$$(1-x^2)y_{n+2}-(2n+3)xy_{n+1}-(n+1)^2y_n=0$$

Solution:

We have

$$y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\therefore \quad (1-x^2)y^2=(\sin^{-1}x)^2$$

Differentiating with respect to both sides,

$$(1-x^2)2yy_1-2xy^2=2\frac{\sin^{-1}x}{\sqrt{1-x^2}}=2y$$

$\therefore \quad (1-x^2)y_1 - xy = 1$

Differentiating with respect to x

$$(1-x^2)y_2 - 2xy_1 - xy_1 - y = 0$$

$$(1-x^2)y_2 - 3xy_1 - xy_1 - y = 0$$

Applying Leibnitz term by term, we get

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n-1} + \frac{n(n-1)}{2!}(-2)y_n\right]$$

$$-3[xy_{n+1} + n\cdot 1\cdot y_n] - y_n = 0$$

$$(1-x^2)y_{n+2} - x(2n-3)y_{n+1} - (n+1)^2 y_n = 0$$

Example 20: If $y = \sec^{-1} x$ then show that

$$x\,(x^2-1)\,y_{n+2} + \left[(2+3n)\,x^2 - n + 1\right]\,\mathrm{y}_{n+1} + n\,(3n+1)\,xyn + n^2(n-1)\,\mathrm{y}_n - 1 = 0$$

Solution:

$$y = \sec^{-1} x$$

Differentiating with respect to x

$$y_1 = \frac{1}{x\sqrt{x^2-1}}$$

$$\therefore \quad \mathrm{x}^2(\mathrm{x}^2-1)y^2{}_1 = 1$$

i.e. $\quad (\mathrm{x}^4 - x^2)y^2{}_1 = 1$

Differentiating with respect to x

$$(4\mathrm{x}^3 - 2x)y^2{}_1 + (\mathrm{x}^4 - x^2)2y_1y_2 = 0$$

i.e. $\quad (2\mathrm{x}^2-1)y_1 + (\mathrm{x}^3 - x)y_2 = 0$

i.e. $\quad (\mathrm{x}^3 - x)y_2 + (2\mathrm{x}^2-1)y_1 = 0$

*D*ifferentiating term by term n times using Leibnitz theorem, we get,

$$\left[(x^3-x)y_{n+2} + n(3x^2-1)y_{n+1} + \frac{n(n-1)}{2!}(6x)y_n\right]$$

$$+3\left[\frac{n(n-1)(n-2)}{3!}(6)y_{n-1}\right]$$

$$+\left[(2x^2-1)y_{n+1} + n(4x)y_n + \frac{n(n-1)}{2!}(4)y_{n-1}\right] = 0$$

$$i.e. \left(x^3 - x\right) y_{n+2} + n\left(3x^2 - 1\right)\left(2x^2 - 1\right) y_{n+1} + \left[3n\left(n-1\right)x + 4nx\right] y_n$$
$$+ \left[n\left(n-1\right)\left(n-2\right) + 2n\left(n-1\right)\right] y_{n-1} = 0$$

$$i.e. x\left(x^2 - 1\right) y_{n+2} + \left[\left(2+3n\right)x^2 - \left(n+1\right)\right] y_{n+1}$$
$$+ n\left(3n+1\right) x y_n + n^2\left(n-1\right) y_{n-1} = 0$$

Example 21: If *$y = tan^{-1}x$, then show that.*

$\left(x^2 + 1\right) y_{n+1} + 2nxy_n + n\left(n-1\right) y_{n-1} = 0$ and also show that

$y_n\left(0\right)$ *is* 0, (n-1)! or 4r+3 respectively

Solution:

Step I :

$y = \tan^{-1} x$

$\therefore \qquad y_1 = \dfrac{1}{1+x^2}$

$\therefore \qquad \left(x^2+1\right) y_1 = 1$

Applying Leibtnitz theorem to differentiate n times, we get,

$$\left[\left(x^2+1\right) y_{n+1} + n\left(2x\right) y_n + \frac{n\left(n-1\right)}{2!}\left(2\right) y_{n-1}\right] = 0$$

i.e. $\qquad \left(x^2+1\right) y_{n+1} + 2nxy_n + n\left(n-1\right) y_{n-1} = 0$........................(1)

Step (II) :

Now, $\qquad y_1\left(0\right) = 1$

And $\qquad y_2\left(0\right) = \dfrac{-2x}{\left(1+x^2\right)^2}$

Putting $\qquad y_2 \ \left(0\right) = 0$

$\therefore \quad$ n = 2,3,4,5,6 in (1)

$y_3 = -2 = -\left(-2\right)!$

$y_4 = 0$

$y_5 = 4!$

$y_6 = 0$

$y_7 = -6!$

:

$\therefore \qquad y_n\left(0\right) = 0 n = 2r$

$y_n\left(0\right) = \left(n-1\right) f$ ifn=4r+1

and $\qquad y_n\left(0\right) = -\left(n-1\right)!$ if n=4r+3

Check Your Progress:

1. *If* $y = \sin\left(m\sin^{-1} x\right)$ then show that

$$\left(1\text{-x}^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 - m^2\right)y_n = 0$$

$\left[H\text{int} : \text{Find } y_1 \ then\right.$

$\left.\left(1-x^2\right)y_1^2 = m^2\left(1-y^2\right) \text{ and again differentiate and apply L. theorem}\right]$

2. *If* $y=e^{a\sin^{-1}x}$ then show that

$$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 + a^2\right)y_n = 0$$

3. *If* $y=a\cos(\log x) - b\sin(\log x)$

then show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + \left(n^2+1\right)y_n = 0$$

$\left[H\text{int}: x^2y_2 + xy_1 + y = 0 \rightarrow apply \text{ Leibnitz theorem}\right]$

4. *If* $y=\left(\sin^{-1}x\right)^2$ then show that :

$$\left(1-x^2\right)y_{n+2}(2n+1) - xy_{n+1} - n^2y_n = 0$$

5. *If* $x = \tan(\log y)$ then show that:

$$\left(1-x^2\right)y_{n+2} + \left[2(n+1)x - 1\right]y_{n+1} + n(n+1)y_n = 0$$

$\left[We \text{ have } y=e^{\tan^{-1}x}.....\right]$

6. *If* $y=\sin\left[\log\left(x^2+x+1\right)\right]$, prove that

$$\left(x+1^2\right)y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

$\left[H\text{int} : \text{Find } y_1, \text{ simplify, find } y_2 \text{ simplify and apply leibnitz theorem}\right]$

10.6 LET US SUM UP

In this unit we have learnt

nth order derivative formula

i) $y = \dfrac{1}{ax+b} \Rightarrow y_n = \dfrac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

ii) $y = \log(ax+b) \Rightarrow y_n = \dfrac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n}$

iii) $y = a^{mx} \Rightarrow y_n = a^{mx}(\log a)^n . m^n$

iv) $y = \sin(ax+b) \Rightarrow y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

v) $y = e^{ax}\sin(bx+c) \Rightarrow y_n = r^n e^{ax}\cos(bx+c+n\alpha)$

v) $y = e^{ax}\sin(bx+c) \Rightarrow y_n = r^n e^{ax}\cos(bx+c+n\alpha)$

where $r = \sqrt{a^2+b^2}$

and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

- Leibnatz's theorem

10.7 Unit End Exercise

1. Find nth order derivative of the following functions:

i) $(8x - 7)^9$

ii) Sin(9x+3) + cos(2x+5)

iii) $\text{Cos}^6 2x$

iv) Sin4xsin3x

v) 2sinxcosx

2. If $y = \frac{x^{20}+5x^{19}+7}{x+5}$, find y_{20}.

3. If $y = \frac{3x^{35}+7x^{34}+12}{x+7}$, find y_{35}.

4. Find 5th order derivative of $y = x^4 e^x$.

5. Find 4th order derivative of $y = x^3 \sin x$.

6. If $y = x^n \log x$, then show that $y_{n+1} = \frac{n!}{x}$.

7. If $\log y = \tan^{-1} x$, then show that

(i) $(1+x^2)y_1 = y$

(ii) $(1+x^2)y_n = [1-2(n-1)x]y_{n-1} = (n-1)(n-2)y_{n-2}$

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11

PARTIAL DIFFERENTIATION

UNIT STRUCTURE

11.1 OBJECTIVE

After going through this unit, you will be able to

- Find Partial Differentiation.
- Total Partial derivative
- Euler's theorem
- Approximation and error
- Maxima and Minima

11.2 INTRODUCTION

So far, we have been concerned with a functions of a variable, but in many problems in science and mathematics we have to deal with functions of two or more independent variables.

e.g. the lift L of an aeroplane wing is a function of three independent variables : A, the area of the wing, V, the speed at which the wing is moving; and P the density of the air. The law is $L = Akv^2 p$

In the language of mathematics, if variable u has one definite value for any given values of x,y,z then u is defined as a function of x,y,z.We represent it as
u= f(x,y,z)

Note that u is independent variable and x,y,z are indepedent variables. This relation is written as - $u \rightarrow x, y, z$

11.3 PARTIAL DIFFERENTIAL COEFFICIENTS

The partial derivative of u= f(x,y,z) with respect to x is the oridnary derivative of u with respect to x when y and z are regarded as constant. It is denoted by $\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}$ or f_x.

(To be pronounced as dabba u by dabba x)

Thus, $\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}$ or $f_x = \lim_{x \to o} \frac{f(x+h, y, z) - f(x, y, z)}{h}$

Similarly when we differentiate u with respect to y we keep x and z constant and so on

.

In general, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are also functions of x,y,z, so we can obtain higher ordered partial derivatives of u=f(x,y,z)

e.g $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

And $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$ and so on.

Note :

In general , $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

11.3.1 RULES OF PARTIAL DIFFERENTIATION:

(1) Let,u v be functions of x,y,z Then

$$\frac{\partial}{\partial x}(u \pm v) = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x}$$

(2) $$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}(kv) = k\frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial x}\left(\frac{k}{v}\right)=-\frac{k}{v^2}\cdot\frac{\partial v}{\partial x}$$

11.3.2 Chain Rules:

Chain- rules are to be developed by drawing flow- diagrams.
Study this point carefully.

(1) $Let\ u=f(x,y,z)$ and $x=\phi_1$ $t=\phi_2(t)$, $z=\phi_3(t)$

[i.e. u is a function of x,y,z and x,y,z each is a function of only variable t]

(1) Thus , $\therefore$ $$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

($\because$ u is a function of only variable t,

$\therefore$ we wirte total derivative $\frac{du}{dt}$ and not $\frac{\partial u}{\partial t}$)

e.g. if $u=x^2+y^2+z^2, x=t, y=t^2, z=t^3$

then $u\rightarrow x,y,z\rightarrow t$

$$\frac{\partial u}{\partial t}=(2x)\cdot 1+(2y)(2t)+(2z)3t^2$$

(2) $If\ u=f(t)$ and $t=\varphi_1(x,y,z)$

i.e. $u\rightarrow t \rightarrow x,y,z$

then $$\frac{\partial u}{\partial x}=\frac{du}{dt}\cdot\frac{\partial t}{\partial x}$$

e.g. $u=t^3, t=x^2+y^2+z^2$

then $$\frac{\partial u}{\partial x}=3t^2\cdot 2x=6xt^2$$

3) $If\ u=f(x,y,z)$, $x=\phi_1(r,s)$,

$y=\phi_2(r,s)$,

$z=\phi_3(r,s)$,

then the flow diagram becomes,

i.e. $u\rightarrow x,y,z \rightarrow r,s$

If we want $\frac{\partial u}{\partial s}$ then it is given by

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

e.g. $u=x^2+y^2+z^2, x=r+s+t,\ y=s^2+t^2, z=t^3$

then $\frac{\partial u}{\partial s} = 2x \cdot 1 + 2y \cdot 2s + 2z \cdot 0 = 6x + 4ys$

11.4 TOTAL DIFFERENTIATION

In Partial differentiation of a function of two or more variables , only one variable varies. But in total differentiation, increments are given in all the variables.

*L*et $z = f(x, y)$

Let ∂z be the increment in z corresponding to the increments ∂x and ∂y in x and y respectively

Replace ∂ by δ only

Then $z + \delta z = f\left(x + \delta x, y + \partial y\right)$

$\therefore \quad \delta z = f\left(x + \partial x, y + \partial y\right) - f(x, y)$

$\therefore \quad \delta z = f\left(x + \partial x, y + \partial y\right) - f\left(x, y + \partial y\right) + f\left(x, y + \partial y\right) - f(x, y)$

or $\delta z = \left[\frac{f\left(x + \partial x, y + \partial y\right) - f\left(x, y + \partial y\right)}{\partial x} \cdot \partial x\right] + \left[\frac{f\left(x, y + \partial y\right) - f(x, y)}{\partial y}\right]$

∂y

Taking limits as $\delta x \to 0, \delta y \to 0$ we get $\delta z = \frac{\delta f}{\delta x} \cdot d x + \frac{\delta f}{\delta y} \cdot dy$

$d z$ is called as total differential of z

Let us see some Corollaries:

(1) Let

$u = f(x, y, z)$ and $x = \varphi_1(t)$, $y = \varphi_2(t)$, $z = \varphi_3(t)$

[i.e. u is function of x,y,z and x,y,z each is a function of only one variable t.]

Thus,

$u = \to x,y,z \to t$

$\therefore \quad \frac{d u}{d t} = \frac{\partial u}{\partial x} \cdot \frac{d x}{d t} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$\Big(\because$ u is a function of only one variable t,

$\therefore$ We write total derivative $\frac{d u}{d t}$ and not $\frac{\partial u}{\partial t}\Big)$

e.g. If u= $x^2 + y^2 + z^2$, $y = t^2$, $z = t^3$

then u $\to$ x, y, z $\to$ t

$$\frac{du}{dt} = (2x)\cdot 1 + (2y)\,(2t)\; + \;(2z)\; 3t^2$$

(2) Let u= f(x, y) and φ (x, y) = 0

$\because$ φ (x, y) = 0, y can be regarded as

a function of x and hence flow-diagram is

$$u \;\rightarrow\; x, y \;\rightarrow\; x$$

$\therefore$ $$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot 1 + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

e.g. if $u = x^2 + y^2$

and $x^3 + y^3 + 3xy = 4$ then to find $\frac{du}{dx}$

$\because$ $x^3 + y^3 + 3xy = 4$

Differentiate with respect to x,

$$3x^2 + 3y^2 \frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

$\therefore$ $$\frac{dy}{dx} = -\frac{(x^2 + y)}{(x + y^2)}$$

and $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 2x + 2y\left[-\frac{x^2 + y}{x + y^2}\right]$$

$$= \frac{2x\,(x + y^2) - 2y(x^2 + y)}{(x + y^2)} = \frac{2\left[x^2 - y^2 \neq xy^2 - x^2\, y\right]}{(x+y^2)}$$ 3)

If f (x,y) = 0 then to find $\frac{dy}{dx}$\

[*This* result is a special case of result (4)].

Let u = f (x, y) and f (x, y) = 0

$\therefore$ $u \;\rightarrow\; x, y$

and $\because$ $f\,(x, y) = 0$

$\therefore$ $y \;\rightarrow\; x$

$\therefore$ $u \;\rightarrow\; x, y \rightarrow x$

then $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$\because$ u = 0 $\therefore$ $\frac{du}{dx} = 0$

$\therefore$ $$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}$$

or, $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

If $u = f(x, y, z)$

where y and z are all functions of x, then we have

$$u \rightarrow x, y, z \rightarrow x \quad \text{and}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx},$$

Also note that if $f(x, y, z) = 0$

then $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx} = 0$$

4) *If* $f(x, y) = 0$ then to find $\frac{d^2 y}{dx^2}$,

We use the following notations :

$$p = \frac{\partial f}{\partial x}, \; q = \frac{\partial f}{\partial y}, \; r = \frac{\partial^2 f}{\partial x^2}, \; s = \frac{\partial^2 f}{\partial x \partial y}, \; t = \frac{\partial^2 f}{\partial y^2}$$

$\because \quad f(x, y) = 0$

$$\therefore \quad \frac{d^2 y}{dx^2} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\left(\frac{p}{q}\right) \qquad (\text{Result } 5)$$

$$\therefore \quad \frac{d^2 y}{dx^2} = -\left[\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}\right] \text{..................(i)}$$

$$\because \quad p, q \rightarrow x, y \rightarrow x$$

$$\therefore \text{ and } \quad \frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx} =$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\left(\frac{p}{q}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial x}\left(\frac{p}{q}\right) = r - s \cdot \frac{p}{q} = \frac{rq - sp}{q}$$

$$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\left(\frac{p}{q}\right)$$

$$= \frac{\partial^2 f}{\partial x \cdot \partial y} - \frac{\partial^2 f}{\partial y^2} \cdot \frac{p}{q}$$

$$s - t \cdot \frac{p}{q}$$

$$= \frac{sq - pt}{q}$$

$\therefore$ from (i), $$\frac{d^2 y}{d x^2} = -\frac{1}{q^2}\left[q\left\{\frac{rp - sp}{q}\right\} - p\left\{\frac{sq - pt}{q}\right\}\right] =$$

$$= -\frac{1}{q^3}\left[q^2 r - 2pqs + p^2 t\right]$$

11.5 SOME ADDITIONAL RESULTS

Partial Diffferentiation applied to :

(1) Brackets : $\frac{\partial}{\partial x}\left[f(x, y, z)\right]^n = n\left[f(x, y, z)\right]^{n-1}\frac{\partial f}{\partial x}$

(2) Trignometric function : $\frac{\partial}{\partial x}\sin\left[f(x, y, z)\right] = \cos\left[f(x, y, z)\right]\cdot\frac{\partial f}{\partial x}$

(3) Expotential Function : $\frac{\partial}{\partial x}a^{[f(x, y, z)]} = a^{[f(x, y, z)]}\cdot \log a\cdot\frac{\partial f}{\partial x}$

(4) Log - function : $\frac{\partial}{\partial x}\left[\log\left\{f(x, y, z)\right\}\right] = \frac{1}{f\,(x, y, z)}\frac{\partial f}{\partial x}$

(5) Inverse Trignometric function :

$$\frac{\partial}{\partial x}\sin^{-1}\left[f(x, y, z)\right] = \frac{1}{\sqrt{1 - f^2\,(x, y, z)}}\cdot\frac{\partial f}{\partial x}$$

Note : (1) In general, $\frac{\partial u}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial u}}$

e.g. if $x = r\cos\theta$ and $y = r\sin\theta$

then $\left(\frac{\partial x}{\partial r}\right) = \cos\theta$

and since , $x^2 + y^2 = r^2$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

from (i) and (ii),

$$\frac{\partial x}{\partial r} \neq \frac{1}{\frac{\partial r}{\partial x}}$$

(2) When we write

$$u \rightarrow x, y, z$$

It means u depeds on x, y, z and x,y,z are independent among themselves.

EXAMPLES

TYPE – I

NOTE :

Problem in this type are based on direct differentiation

(1) First find dependent and independent variables.

(2) Use the necessary formulae.

Examples 1 :

If $z = x^y + y^x$ then show that $\frac{\partial^2 z}{\partial x \, \partial y} = \frac{\partial^2 z}{\partial y \, \partial x}$.

Solution: $z = x^y + y^x \qquad \therefore \; z \to x, y$

$$\frac{\partial z}{\partial y} = yx^{y-1} + y^x \cdot \log x \qquad ....(i)$$

and $$\frac{\partial z}{\partial y} = x^y \log x + x\, y^{x-1} \qquad ....(ii)$$

Differentiating (i) partially with respect to y,

$$\frac{\partial^2 z}{\partial x \, \partial y} = \left(y\, x^{y-1} \cdot \log x + x^{y-1}\right) + \left(y^x \cdot \frac{1}{y} + \log y \cdot xy^{x-1}\right)$$

$$= y\, x^{y-1} \cdot \log x + x^{y-1} + y^{x-1} + xy^{x-1} \cdot \log y ....(iii)$$

Differentiating (i) partially with respect to x,

$$\frac{\partial^2 z}{\partial x \, \partial y} = x^y \cdot \frac{1}{x} + yx^{y-1} \cdot \log x + 1 \cdot y^{x-1} + xy^{x-1} \cdot \log y ....(iv)$$

From (iii) and (iv)

$$\frac{\partial^2 z}{\partial x \, \partial y} = \frac{\partial^2 z}{\partial y \, \partial x}$$

Examples 2: *If* $u = \log (x^3 + y^3 + z^3 - 3xyz)$ then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$$

Solution:

Note that u is a function of x, y, z

i.e. $u \to x, y, z$

$$u = \log (x^3 + y^3 + z^3 - 3xyz)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)} (3x^2 - 3yz)$$

[*see* the rule of partial Differentiating applied to log function]

and similarly

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)} (3y^2 - 3xz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}\,(3z^2 - 3xy)$$

$$\therefore\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\,(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3\,(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x + y + z}$$

Note that :

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x + y + z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x + y + z}\right)$$

$$= \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} = \frac{-9}{(x + y + z)^2}$$

Example 3: *If* $v = \left(1 - 2xy + y^2\right)^{-1/2}$ then show that

(i) $x\,\frac{\partial v}{\partial x} - y\,\frac{\partial v}{\partial y} = y^2\,v^3$ and

(ii) $\frac{\partial}{\partial x}\left\{\left(1 - x^2\right)\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\,\frac{\partial v}{\partial y}\right\} = 0$

Solution: (i) $v \rightarrow x, y$

We write $v^2 = 1 - 2xy + y^2$

Differentiate partially with respect to x and y,

(see the rule of Partial Differentiation applied to Brackets).

$$-2\,v^{-3}\,\frac{\partial v}{\partial x} = -2y$$

$$\therefore \qquad \frac{\partial v}{\partial x} = v^3\,y \qquad ......(1)$$

and $\quad -2v^3\,\frac{\partial v}{\partial y} = -2x + 2y$

$$\frac{\partial v}{\partial y} = v^3\,(x - y) \qquad ....(2)$$

from 1 and 2

$$x\,\frac{\partial v}{\partial x} - y\,\frac{\partial v}{\partial y} = xy\,v^3 - yv^3\,(x-y) = y^2\,v^3$$

(ii) $\because \quad \left(1 - x^2\right) \dfrac{\partial v}{\partial x} = \left(1 - x^2\right) yv^3$ from (1)

$$\therefore \quad \frac{\partial}{\partial x}\left[\left(1 - x^2\right) \frac{\partial v}{\partial y}\right] = y \frac{\partial}{\partial x}\left[\left(1 - x^2\right) v^3\right]$$

($\because$ y is constant, and v is a function of x, y)

$$= y\left[-2xv^3 + \left(1 - x^2\right) 3v^2 \frac{\partial v}{\partial x}\right]$$

$$= y\left[-2xv^3 + 3\left(1 - x^2\right) v^2\, y\, v^3\right] \text{ from (1)}$$

$$= yv^3\left[-2x + 3\left(1 - x^2\right) yv^2\right] \quad ....(3)$$

Again, $\dfrac{\partial}{\partial y}\left\{y^2 \dfrac{\partial v}{\partial y}\right\} = \dfrac{\partial}{\partial y}\left\{y^2\, v^3\, (x-y)\right\}$

$$= 3\,v^2 \frac{\partial v}{\partial y}\, x\left(x\,y^2 - y^3\right) + v^3\,(2xy - 3y^2) \quad (\because\ v \to x,y)$$

$$= \left[3\,v^2 \cdot v^3 \cdot (x - y) \cdot \left(xy^2 - y^3\right) + v^3\,(2xy - 3y^2)\right]$$

$$= v^3\, y\left[3\,v^2\,(x - y)\left(xy - y^2\right) + (2x - 3y)\right]$$

$$= v^3\, y\left[3\,v^2\,(x - y)\left(xy - y^2\right)(2x - 3y)\right] \quad ....(4)$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{\left(1 - x^2\right) \frac{\partial v}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial v}{\partial y}\right\}$$

$$= y\,v^3\left[-2x + 3y\left(1 - x^2\right)v^2 + 3\,v^2\, y\left(x - y^2\right) + (2x - 3y)\right] \text{ from (3), (4),}$$

$$= y\,v^3\left[3y\left(1 - x^2\right)\left(x - y^2\right) v^2 - 3y\right]$$

$$= y\,v^3\left[3y\left(1 - x^2 + x^2 - 2xy + y^2\right)v^2 - 3\,y\right]$$

$$= y\,v^3\left[3y\left[v^{-2}\right] v^2 - 3y\right]$$

$$= 0 \qquad \because\ v^2 = 1 - 2\,xy + y^2$$

Example 4 : *If* $u = \log\left(\tan x + \tan y + \tan z\right)$ then show that

$$\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z\, \frac{\partial u}{\partial z} = 2$$

Solution: Here $u \to x, y, z$

(Using the rule of partial diff. applied to log function) we have,

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 y$$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 z$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z \cdot \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x . \sec^2 x + \sin 2y . \sec^2 y + \sin 2z. \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\left(\tan x + \tan y + \tan z\right)}{\left(\tan x + \tan y + \tan z\right)} = 2$$

$$\because \quad \sin 2x. \sec^2 x = 2 \sin x . \cos x . \frac{1}{\cos^2 x} = 2 \tan x$$

Examples 5 : *If* $\theta = t^n , e^{-r^{2/4t}}$ then find the value of n show that

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

Solution: We have $\theta \rightarrow r, t$

(To simplify the expression, we take log).

We have, $\log \theta = n \log t - \frac{r^2}{rt}$

Diff. partially with respect to r,

$$\frac{1}{\theta} \frac{\partial \theta}{\partial r} = \frac{-2 r}{4 t}$$

$$\therefore \quad \frac{\partial \theta}{\partial r} = - \frac{r \theta}{2}$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = - \frac{r^2 \theta}{2 t}$$

Diff. partially with respect to r,

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = - \frac{1}{2}\left[3 r^2 \theta + r^3 \frac{\partial \theta}{\partial r}\right] =$$

$$- \frac{1}{2 t}\left[3 r^2 \theta - \frac{r^r \theta}{2 t}\right] \quad \text{...... from (1)}$$

$$\therefore \quad \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = - \frac{1}{2}\left[3 \theta - \frac{r^2 \theta}{2 t}\right] \quad \text{........ (2)}$$

Again diff. given relation with respect to t partially ,

$$\frac{1}{\theta} \frac{\partial \theta}{\partial t} = \frac{n}{t} + \frac{r^2}{4} \cdot \frac{1}{t^2}$$

$$\therefore \quad \frac{\partial \theta}{\partial t} = \theta \left[\frac{n}{t} + \frac{r^2}{4 t^2}\right]$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t},$$

From (2) and (3) we get,

$$\frac{n\theta}{t}+\frac{r^2\theta}{4t^2}=-\frac{3}{2}\frac{\theta}{t}+\frac{r^2\theta}{4t^2} \qquad \therefore \ n=-\frac{3}{2}$$

Example 6 : *If* $u(x,t)=Ae^{-gx}\cdot\sin\left(nt-gx\right)$

and if $\frac{\partial u}{\partial x}=u\frac{\partial^2 u}{\partial x^2}$ *t*hen show that $g=\sqrt{\frac{n}{2u}}$

Solution: $u\rightarrow x,t$

We have, $\frac{\partial u}{\partial t}=Ae^{-gx}\cdot\cos\left(nt-gx\right)\cdot n$

$=Ane^{-gx}\cos\left(nt-gx\right)\left(\because \ x \text{ is to be kept constant}\right)$........(1)

Again diff. u partially with respect to x, we get,

$$\frac{\partial u}{\partial x}=A\left[-g\,e^{-gx}\cdot\sin(nt-gx)-g\cdot e^{-gx}.\cos\left(nt-gx\right)\right]$$

$=-Ag\,e^{-gx}\left[\sin(nt-gx)+\cos\left(nt-gx\right)\right]$

(Rule of partial differentitation applied to product)

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}=-Ag\left[-g\,e^{-gx}\sin(nt-gx)+\cos\left(nt-gx\right)\right]$$

$e^{-gx}\left[-g\cdot\cos(nt-gx)+g\sin\left(nt-gx\right)\right]$

$=+Ag^2e^{-gx}\left[2\cos(nt-gx)\right]$

$$\because \qquad \frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}$$

$\therefore$ From (1) and (2)

$(An.e^{-gx}.\cos(nt-gx)=\mu.2.Ag^2.e^{gx}.\cos(nt-gx)$

$$n=2g^2\mu \qquad \therefore \ g=\sqrt{\frac{n}{2\mu}}$$

Example 7 : *If* $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$, then find $\frac{\partial^2 u}{\partial x\,\partial y}$.

Solution: $u\rightarrow x,y.$

We have , $\frac{\partial u}{\partial y}=x^2\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)-y^2\cdot\frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)-2y\tan^{-1}\frac{x}{y}$

$$=\frac{x^3}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}+\frac{xy^2}{x^2+y^2},$$

$$= \frac{x^3 + xy^2}{x^2 + y^2} - 2y \cdot \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1} \frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[2 - 2y \tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2 - y^2}{x^2+y^2}$$

Check your progress:

1) If u, (x + y) = $x^2 + y^2$ then show that :

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\left[Hint \;\; u = \frac{x^2 + y^2}{x + y} \;\; \therefore \; u \to x, y \text{ find } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \; .... \right]$$

(2) Find the value of n so that u= r^n (3 $\cos^2 \theta$ - 1) satisfy the equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin \theta}\left(\sin \theta \frac{\partial u}{\partial \theta}\right) = 0$$

$\left[\text{Hint : u} \to \text{r}, \theta, \text{ Find } \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta} \; .... \right]$ Ans. n = 2, -3.

(3) If u = e^{xyz} then show that

$$\frac{\partial^3 u}{\partial x \, \partial y \, \partial z} = \left(1 + 3xyz + x^2 y^2 z^2\right) \cdot e^{xyz}$$

$$\left[\text{First find } \frac{\partial u}{\partial z} \text{ then } \frac{\partial^2 u}{\partial x \, \partial z} \; and \; \frac{\partial^3 u}{\partial x \cdot \partial y \, \partial z}\right]$$

(4) *If* $v = \frac{c}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}$, then prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$

$\left[Hint \; : \text{Take log}, \; \therefore \; \log v = \log c - \frac{1}{2}\log t - \frac{x^2}{4a^2 t}\right]$ $v \to x, t$

(Find $\frac{\partial v}{\partial t}$ and $\frac{\partial^2 v}{\partial x^2}$, apply the rule of P.D. applied to log function)

(5) Find $\frac{\partial^2 u}{\partial y \, \partial z}$ where u = log ($x^2 + y^2 + z^2$) Ans. $\frac{-4yz}{(x^2 + y^2 + z^2)^2}$

(6) Verify $\frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial^2 u}{\partial y}$ Where (i) u = log (y sin x + x sin y)

(7) If $u = x^m\ y^n$ then show that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 u}{\partial y\,\partial x^2}$

(8) *If* $u=\log\left(y \sin x + x \sin x\right)$ then show that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial y}$

(9) If $u=\log\sqrt{x^2+y^2+z^2}$ then show that

$$\left(x^2+y^2+z^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$$

$\Big[$ *H* int : $\because \quad e^{2u} = x^2+y^2+z^2 \qquad \therefore\ u \to x,y,z$

$$\therefore\ 2e^{2u}\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial x} = x\,e^{-2u}$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-2u} + x\left(-2e^{-2u}\right)\frac{\partial u}{\partial x} = e^{-2u} - 2xe^{-2e}\cdot x\,e^{-2e}$$

$$= e^{-2u} - 2x^2\,e^{-4u}$$

10) If $u = r^m$, $\quad r = \sqrt{x^2+y^2+z^2}$

then find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$

$\left[\text{Hint : } u=\left(x^2+y^2+z^2\right)^{m/2} \qquad u \to x,y,z\ \therefore\ \text{find } \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\ldots\ldots\right]$

Ans : $m\left(m+1\right) r^m$

TYPE – II

Note :- Here we deal with the the problems of the type u = f (x, y, z) where x, y functions of x, y, z

i.e. $\quad u \to X, Y, Z \qquad \to x, y, z.$

We shall be frequently using the Chain- Rules can be develop drawing the flow- diagram.

Example 8 : *If* $u=f\left(x-y, y-z, z-x\right)$ then show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Solution: Let $\quad X=x-y,\ Y=y-z,\ Z=z-x$

so that $u=f\left(X,Y,Z\right)$

$$u \to X,Y,Z \to x,y,z$$

$\therefore \qquad \dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial X}\cdot\dfrac{\partial X}{\partial x}+\dfrac{\partial u}{\partial Y}\cdot\dfrac{\partial Y}{\partial x}+\dfrac{\partial u}{\partial Z}\cdot\dfrac{\partial Z}{\partial x}$

$$=\frac{\partial u}{\partial X}\cdot(1)+\frac{\partial u}{\partial Y}\cdot(0)+\frac{\partial u}{\partial Z}\text{-1}=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}$$

and $\qquad \dfrac{\partial u}{\partial Y}=\dfrac{\partial u}{\partial X}\cdot\dfrac{\partial X}{\partial y}+\dfrac{\partial u}{\partial Y}\cdot\dfrac{\partial Y}{\partial y}+\dfrac{\partial u}{\partial Z}\cdot\dfrac{\partial Z}{\partial y}$

$\therefore \qquad \dfrac{\partial u}{\partial Y}=\dfrac{\partial u}{\partial X}\cdot(-1)+\dfrac{\partial u}{\partial Y}\cdot(1)+\dfrac{\partial u}{\partial Z}\cdot(0)$

$$=\frac{\partial u}{\partial Y}-\frac{\partial u}{\partial X}$$

Similarly $\quad \dfrac{\partial u}{\partial Z}=\dfrac{\partial u}{\partial X}\cdot\dfrac{\partial X}{\partial z}+\dfrac{\partial u}{\partial Y}\cdot\dfrac{\partial Y}{\partial z}+\dfrac{\partial u}{\partial Z}\cdot\dfrac{\partial Z}{\partial z}$

$$=\frac{\partial u}{\partial X}\cdot(0)+\frac{\partial u}{\partial Y}\cdot(-1)+\frac{\partial u}{\partial Z}\cdot(1)=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}\text{.............}(3)$$

From (1), (2), (3)

$$\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}=0$$

Example 9 : *If* $u=f\left(\dfrac{x^2}{y}\right)$ then prove that

(i) $x\dfrac{\partial u}{\partial x}+2y\left(\dfrac{\partial u}{\partial y}\right)=0$ (ii) $x^2\dfrac{\partial^2 u}{\partial x^2}+3xy\dfrac{\partial^2 u}{\partial x\,\partial y}+2y^2\dfrac{\partial^2 u}{\partial y^2}=0$

Solution:

Let $\qquad t=\dfrac{x^2}{y^2} \qquad$ so that $\quad u=f(t)$

[*Note* that u is a function of only one variable $\dfrac{x^2}{y}$ =t, which in turn is a function of x and y]

$\therefore \qquad u \to t \to x,y \quad$ (see chain rule 2)

$\therefore \qquad \dfrac{\partial u}{\partial x}=\dfrac{du}{dt}\cdot\dfrac{\partial t}{\partial x}=\dfrac{du}{dt}\cdot\dfrac{2x}{y}$

and, $\quad \dfrac{\partial u}{\partial y}=\dfrac{du}{dt}\cdot\dfrac{\partial t}{\partial y}=\dfrac{du}{dt}\cdot\left(\dfrac{-x^2}{y^2}\right)$

$\therefore\ x\dfrac{\partial u}{\partial x}+2y\dfrac{\partial u}{\partial y}=\dfrac{2x^2}{y}\dfrac{du}{dt}-\dfrac{2x^2}{y}\dfrac{du}{dt}=0$

i.e. $x\dfrac{\partial u}{\partial x}+2y\dfrac{\partial u}{\partial y}=0\text{.........................}(1)$

Diff (1) partially with repect to y we get,

$$x\frac{\partial^2 u}{\partial x^2}+1\cdot\frac{\partial u}{\partial x}+2y\frac{\partial^2 u}{\partial x\partial y}=0................(2)$$

and $$x\frac{\partial^2 u}{\partial x\partial x}+2y\frac{\partial^2 u}{\partial y^2}+2\cdot\frac{\partial u}{\partial y}=0................(3)$$

Taking (2)$\times$ x+(3) x y, we get

$$\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+2y\frac{\partial u}{\partial y}+2y^2\frac{\partial^2 u}{\partial y^2}\right)=0$$

$$\because\quad x\frac{\partial u}{\partial y}+2y\frac{\partial u}{\partial y}=0 \qquad \text{from (1)}$$

$$\therefore\quad x^2\frac{\partial^2 u}{\partial x^2}+3xy\frac{\partial^2 u}{\partial x\partial y}+2y^2\frac{\partial^2 u}{\partial y^2}=0$$

Example 10: *If* $(\cos x)^y=(\sin y)^x$ then find $\frac{dy}{dx}$

Solution:

Taking logs, we get,

y log cos x = x log sin y

Let f (x,y) =y log cos x -xlog sin y=0

$$\therefore\quad \frac{dy}{dx}=-\frac{df/dx}{df/dy}$$

Now, $$\frac{\partial f}{\partial x}=y\cdot\frac{1}{\cos x}(-\sin x)-\log\sin y$$

$$=-y\tan x-\log\sin y$$

and $$\frac{\partial f}{\partial y}=\log\cos x-x\frac{1}{\sin y}\cos y=\log\cos x-x\cot y$$

$\therefore$ From (1) $$\frac{dy}{dx}=\frac{y\tan x+\log\sin y}{\log\cos x-x\cot y}$$

Example 11: *If* $y^x+x^y=(x+y)^{x+y}$ then find $\frac{dy}{dx}$

Solution:

Let $f(x+y)=(x+y)^{x+y}-y^x-x^y=o$

$$\therefore\quad \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}..................(1)$$

Now, $$\frac{\partial f}{\partial x}=(x+y)^{(x+y)}\cdot[1+\log(x+y)]-y^x\cdot\log y-y\,x^{y-1}$$

and $$\frac{\partial f}{\partial y}=(x+y)^{(x+y)}[1+\log(x+y)]-xy^{x-1}\,x^y\cdot\log x$$

$\therefore$ From (1), $\dfrac{d\ y}{d\ x} = \dfrac{-\left\{(x+y)^{(x+y)}\ [1+\log\ (x+y)]-y^x\ \log y - yx^{y-1}\right\}}{\left\{(x+y)^{(x+y)}\ [1+\log\ (x+y)]-xy^{x-1}-x^y\cdot\log x\right\}}$

Example 12: Prove that at a point of the surface $x^x\ y^y\ z^z = c$

$$\text{where } x=y=z,\ \frac{\partial^2 z}{\partial x\,\partial y} = -\left(x\log ex\right)^{-1}$$

Solution:

$\left(\text{From the expression } \dfrac{\partial^2 z}{\partial x\,\partial y} \text{ it is clear that } z \to x,y\right)$

Taking logs

x log x+y log y+z log z =log c

Different with respect to y partially, (i.e. keeping x constant)

$$0+\log y + y\cdot\frac{1}{y}\ \left(\log z + z\cdot\frac{1}{z}\right)\frac{\partial z}{\partial y} =$$

$$\frac{\partial^2 z}{\partial x} = -\frac{(1+\log y)}{(1+\log z)}$$

Diff. with respect to x partially, we get,

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(1+\log y)\left[\frac{-1}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$= \frac{(1+\log y)}{z\ (1+\log z)^2}\ \frac{\partial z}{\partial x}$$

Now, we can show (as in (1) that $\dfrac{\partial z}{\partial x} = -\dfrac{(1+\log x)}{(1+\log z)}$

$\therefore$ From (2), $\dfrac{\partial^2 z}{dx\,dy} = -\dfrac{(1+\log x)(1+\log y)}{z\ (1+\log z)}$

At x=y=z, $\dfrac{\partial^2 z}{dx\,dy} = -\dfrac{(1+\log x)^2}{x\ (1+\log x)^3} = -\dfrac{1}{x\ (1+\log x)}$

$$= -\frac{1}{x\ (\log e + \log x)} = -\frac{1}{x\ \log\ (e\,x)}$$

$$= \ -\left[x\ \log\ (ex)\right]^{-1}$$

Check your progress:

(1) *If* $z=f\ (x,y,u,v)$ where u,v are functions of x,y then prove that

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$ and write corresponding formulae for

$\frac{\partial z}{\partial y}$. $\left[\text{Hint : } z \to x,y,u,v \to x, y \quad \frac{\partial z}{\partial x} = ...\right]$

(2) If v=f $\left(x^2+y^2+z^2\right)$ then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 4\left(x^2+y^2+z^2\right) f'\left(x^2+y^2+z^2\right) + 6f\left(x^2+y^2+z^2\right)$$

$\left[H\text{int : Let } x^2+y^2+z^2 = u\right.$

$\therefore$ $v \to u \to x,y,z$

$\therefore$ $\left.\frac{\partial v}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial u}{\partial x} \text{ and proceed}\right]$

11.6 TYPE - III VARIABLE TO BE TREATED AS CONSTANT

Notations : $\left(\frac{\partial u}{\partial x}\right)$ means partial derivative of u with respect to x keeping Y constant.

To find $\left(\frac{\partial u}{\partial x}\right)_y$ we must have an equation in u, x and y only.

Example 13: *If* x=r cos θ, y=r sin θ then show that

$$\left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = x^2+y^2,$$

Where surffixes denote variables kept constant

Solution:

$\because$ x=r cos θ, y=r sin θ

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos_\theta, \left(\frac{\partial y}{\partial r}\right)_\theta = \sin\ \theta.$$

$$\therefore\left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = \left[x\cos\theta + y\sin\theta\right]^2 = \left[r\cos^2\theta + r\sin^2\theta\right]^2$$

$$= r^2 = x^2+y^2$$

Example 14: If u=*l*x+my, v=mx-*l*y, then show that:

(i) $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{\iota^2}{\iota^2+m^2}$ and

(ii) $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{\iota^2}{\iota^2+m^2}$

Solution: We have

$$u=l\,x+my$$

$$v=mx-l\,y$$

(i) $\because$ u=l x+my

$\therefore$ $\left(\frac{\partial u}{\partial x}\right)_y = l$ ....(1)

(ii) To find $\left(\frac{\partial x}{\partial u}\right)_v$ we must have relation between x,u and v

Eliminating y from the given relations, we get,

$$lu+mv=\left(l^2+m^2\right)x$$

$$x=\frac{lu+mv}{l^2+m^2}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{l^2+m^2} \qquad ....(2)$$

From (1) and (2)

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$$

(iii) $\because$ v= mx-ly ....(3)

$\therefore$ Diff with respect to v keeping x constant,

$$l=0-l\left(\frac{\partial y}{\partial v}\right)_x$$

$\therefore$ $\left(\frac{\partial y}{\partial y}\right)_x = \frac{1}{l}$

(iv) To find $\left(\frac{\partial v}{\partial y}\right)_u$ $,we\ e\lim inate$ x from given relations,

i.e. $mu-lv=\left(l^2+m^2\right)y$

$\therefore$ $0-l\left(\frac{\partial v}{\partial y}\right)_u = \left(l^2+m^2\right)\cdot 1$

$\therefore$ $\left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2+m^2}{l}$

$\therefore$ From (4), (5)

$$\left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$$

Example 15: *If* $f(x,y,z) = 0$ then show that $\left(\frac{\partial z}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial z}\right)_y}$

Solution:

Here we use the result that if $f(x,y) = 0$

then $$\frac{\partial y}{\partial x}_y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

(i) Here $f(x,y,z)$=0. When y is kept constant,

We have, $$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial z}$$

(ii) And $$\left(\frac{\partial x}{\partial z}\right)_y = -\frac{\partial f/\partial z}{\partial f/\partial x}$$

From (1) and (2) $$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{1}{\left(\frac{\partial x}{\partial z}\right)_y}$$

Example 16: *If* $x=\frac{\cos\theta}{u}$, $y=\frac{\sin\theta}{u}$, evaluate

$$\left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial u}{\partial x}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x$$

Solution

(i) $\because$ $x=\frac{\cos\theta}{u}$

$\therefore$ $$\left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2} \text{................(1)}$$

(ii) $\therefore$ $$y = -\frac{\sin^2\theta}{u^2}$$

$\therefore$ $$\left(\frac{\partial y}{\partial u}\right)_\theta = \frac{\sin^2\theta}{u} \text{....................(2)}$$

(iii) To find $\left(\frac{\partial u}{\partial u}\right)_\theta$, we eliminate θ from given relations, i.e. $$x^2 + y^2 = \frac{1}{u^2}$$

or $$u^2 = \frac{1}{x^2 + y^2} \text{..........................}(A)$$

$$\therefore \quad 2u\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{\left(x^2 + y^2\right)^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = \frac{-x}{u\left(x^2 + y^2\right)^2} \text{..................}(3)$$

(iv) and again $$u^2 = \frac{1}{x^2 + y^2}$$

$$\therefore \quad 2u\left(\frac{\partial u}{\partial y}\right)_x = \frac{-2y}{\left(x^2 + y^2\right)^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial y}\right)_y = \frac{-y}{u\left(x^2 + y^2\right)^2} \text{.....................}(4)$$

From (1), (2), (3), (4) we get,

$$\text{Required expression} = \left(-\frac{\cos\theta}{u^2}\right)\left(\frac{-x}{u\left(x^2 + y^2\right)^2}\right) - \left(\frac{\sin\theta}{u^2}\right)\left(\frac{-y}{u\left(x^2 + y^2\right)^2}\right)$$

$$= \frac{x\cos\theta + y\sin\theta}{u^3\left(x^2 + y^2\right)^2}$$

but $$x\cos\theta + y\sin\theta = \frac{\cos^2\theta + \sin^2\theta}{u} = \frac{1}{u}$$

$$\text{Re}\mathit{quired}\ \text{expression} = \frac{1}{u^4\left(x^2 + y^2\right)^2}$$

$$= \frac{1}{u^4} \cdot u^4 = 1 \qquad \text{(from A)}$$

Example 17: *If* $x+y+z+u+v=a$, $x^2+y^2+z^2+u^2+v^2=b^2$, where a,b are constants, prove that

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} \cdot \left(\frac{\partial x}{\partial u}\right)_{v,z} = \left(\frac{\partial v}{\partial y}\right)_{x,z} \cdot \left(\frac{\partial y}{\partial v}\right)_{u,z}$$

Solution:

To find $\left(\frac{\partial u}{\partial x}\right)_{y,z}$ *we have* to eliminate v from the given equations and as this process will have to be repeated four times, we proceed in the following way.

Let $x+y+z+u+v=a$ ....(1)

$x^2+y^2+z^2+u^2+v^2=b^2$..............(2)

differentiating with respect to x partially keeping y,z as constants, we get,

$$1+\left(\frac{\partial u}{\partial x}\right)_{y,z}+\left(\frac{\partial v}{\partial x}\right)_{y,z}=0 \quad ....(3)$$

and $$2x+2u\left(\frac{\partial u}{\partial x}\right)_{y,z}+2y\left(\frac{\partial v}{\partial x}\right)_{y,z}=0 \quad ....(4)$$

Solving the equations (3), (4) for $\left(\frac{\partial u}{\partial x}\right)_{y,z}$ by Cramer's Rule we get

$$\left(\frac{\partial u}{\partial x}\right)_{y,z}=\frac{v-x}{u-v} \quad ....(5)$$

Similarly differentiating (1), (2) partially with respect to y keeping x, z as constants, we have

$$1+\left(\frac{\partial u}{\partial y}\right)_{x,z}+\left(\frac{\partial v}{\partial y}\right)_{x,z}+2v\left(\frac{\partial v}{\partial y}\right)_{x,z}=0 \quad ....(7)$$

$$ey+ex\left(\frac{\partial u}{\partial y}\right)_{x,z}+2v\left(\frac{\partial v}{\partial y}\right)_{x,z}=0$$

Solving (6), (7) for $\left(\frac{\partial v}{\partial y}\right)_{x,z}$ we get

$$\left(\frac{\partial v}{\partial y}\right)_{x,z}=\frac{y-u}{u-v} \quad ....(8)$$

Similarly differentiating (1), (2) partially with respect to u treating v,z as constant we get,

$$\left(\frac{\partial x}{\partial u}\right)_{u,z}+\left(\frac{\partial u}{\partial u}\right)_{v,z}+1=0 \quad ....(9)$$

and $$2x\left(\frac{\partial x}{\partial u}\right)_{v,z}+2y\left(\frac{\partial y}{\partial u}\right)_{v,z}+2u=0 \quad ....(10)$$

solving (9), (10) for $\left(\frac{\partial x}{\partial u}\right)_{v,z}$ *we* get,

$$\left(\frac{\partial x}{\partial u}\right)_{v,z}=\frac{y-u}{x-y} \quad ....(11)$$

Similarly differentiating (1), (2) partially with respect to v where u,z, are kept constants, we get,

$$\left(\frac{\partial x}{\partial v}\right)_{u,z} + \left(\frac{\partial y}{\partial v}\right)_{u,z} + 1 = 0 \qquad ..... (12)$$

$$2x\left(\frac{\partial x}{\partial v}\right)_{u,z} + 2y\left(\frac{\partial y}{\partial v}\right)_{u,z} + 2v = 0 ....(13)$$

Solving equations (12), (13) for $\left(\frac{\partial y}{\partial v}\right)_{u,z}$ we get,

$$\left(\frac{\partial y}{\partial v}\right)_{u,z} = \frac{v-x}{x-y} \qquad ....(14)$$

From (5), (11) and from (8), (14) we get,

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} \cdot \left(\frac{\partial x}{\partial u}\right)_{v,z} = \frac{v-x}{u-v} \cdot \frac{y-u}{x-y} = \left(\frac{\partial v}{\partial y}\right)_{x,z} \cdot \left(\frac{\partial y}{\partial v}\right)_{u,z}$$

Example 18: If $u = x^2 + y^2$ and $x = s + 3t,\ y = 2s - t$. Find $\frac{\partial^2 u}{\partial t^2}$.

Solution: We have $u = x^2 + y^2$

$$\therefore \qquad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

Now, $u \to x, y \to s, t$

$$\therefore \qquad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x)\ (1) + (2y)\ (2)$$

$$= 2x + 4y$$

Now, $$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) = \frac{\partial}{\partial s}(2x + 4y)$$

$$= 2\frac{\partial x}{\partial s} + 4\frac{\partial y}{\partial s}$$

$$= 2 \times 1 + 4 \times 2 = 10$$

And $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} ....(i)$$

$$= 2x \times 3 + 2y\ (-1)$$

$$= 6x - 2y$$

And $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}(6x - 2y)$$

$$= 6\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t} x_2$$

= 6(3)-2(-1)=20 ....(ii)

Check Your Progress-

(1) *If* x=r cos θ, y= r sin θ then show that $\left(\frac{\partial y}{\partial r}\right)_\theta \left(\frac{\partial y}{\partial r}\right)_\theta = 1$

(2) If $\varphi\ (x,y,z) = 0$ then show that $\left(\frac{\partial z}{\partial y}\right)_x \cdot \left(\frac{\partial x}{\partial z}\right)_y \cdot \left(\frac{\partial y}{\partial x}\right)_z = -1$

(3) If u=ax+by, v=bx-ay, show that

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = 1$$

11.7 LET US SUM UP

In this chapter we have learn Application of Differential equation like-
Partial Derivative of 1st order and 2nd order
Total differentiation
Euler's Theorem
Approximation and error formula
Maxima and Minima of the function.

11.8 UNIT END EXERCISE

1. *Find* $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ if $u = e^{r\cos\theta}.\cos(r\sin\theta).$

2. *Find* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ if $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right).$

3. If $u = (1-2xy+y^2)^{-1/2}$ then prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3.$

4. If $u = (1-2xy+y^2)^{-1/2}$ then prove that $\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = 0.$

5. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then prove that $\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}.$

6. If $u = \log(x^3+y^3+z^3-xyz)$, then prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{9}{(x+y+z)^2}.$

7. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u.$

8. If $u = \log\left(\frac{x^4 + y^4}{x + y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$

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MEAN VALUE THEOREMS 12

Unit Structure

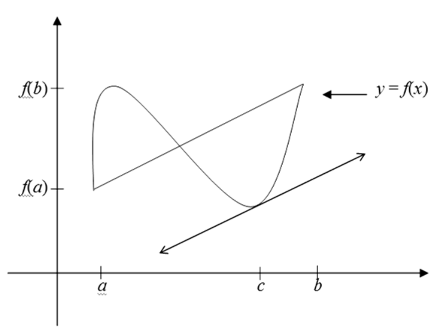
12.0 OBJECTIVES:

After going through this chapter you will be able to:

- State and prove three mean value theorems (MVT): Rolle's MVT, Lagrange's MVT and Cauchy's MVT.

12.1 INTRODUCTION:

The **Mean Value Theorem** is one of the most important theoretical tools in Calculus. Let us consider the following real life event to understand the concept of this theorem: If a train travels 120 km in one hour, then its average speed during is 120 km/hr. The car definitely either has to go at a constant speed of 120 km/hr during that whole journey, or, if it goes slower (at a speed less than 120 km/hr) at a moment, it has to go faster (at a speed more than 120 km/hr) at another moment, in order to end up with an average speed of 120 km/hr. Thus, the Mean Value Theorem tells us that at some point during the journey, the train must have been traveling at exactly 120 km/hr. This theorem form one of the most important results in Calculus. Geometrically we can say that MVT states that given a continuous and differentiable curve in an interval $[a, b]$, there exists a point $c \in [a, b]$ such that the tangent at c is parallel to the secant joining $(a, f(a))$ and $(b, f(b))$.



12.1.1 Rolle's Theorem:

If f is a real valued function such that (i) f is continuous on $[a, b]$, (ii) f is differentiable in (a, b) and (iii)$f(a) = f(b)$ then there exists a point $c \in (a, b)$ such that $f'(c) = 0$

Geometrical Interpretation of Rolle's theorem:

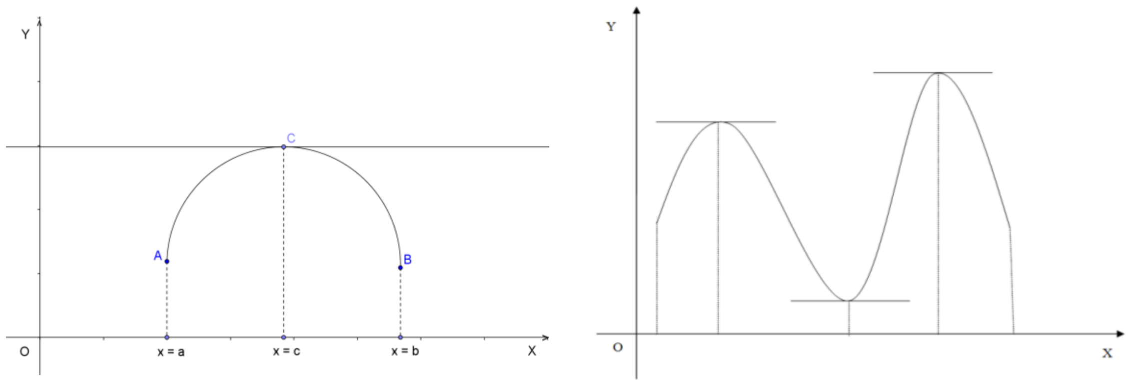


Fig 12.1

We know that $f'(c)$ is the slope of the tangent to the graph of f at $x = c$. Thus the theorem simply states that between two end points with equal ordinates on the graph of f, there exists at least one point where the tangent is parallel to the X axis, as shown in the

Fig 12.1. After the geometrical interpretation, we now give you the algebraic interpretation of the theorem.

Algebraic Interpretation of Rolle's Theorem:

We have seen that the third condition of the hypothesis of Rolle's theorem is that $f(a) = f(b)$. If for a function f, both $f(a)$ and $f(b)$ are zero that is a and b are the roots of the equation $f(x) = 0$, then by the theorem there is a point c of (a, b), where $f'(c) = 0$, which means that c is a root of the equation $f'(x) = 0$.

Thus Rolle's theorem implies that between two roots a and b of $f(x) = 0$ there always exists at least one root c of $f'(x) = 0$ where $a < c < b$. This is the algebraic interpretation of the theorem.

Example 1: Verify Rolle's Theorem for the following:

(1) x^2 in $[-1, 1]$ (2) x^2 in $[1, 3]$

Solution: (1) Let $f(x) = x^2, x \in [-1, 1]$

As $f(x)$ is a polynomial in x, it is continuous and differentiable everywhere on its domain. Also $f(-1) = f(1) = 1$

$\therefore$ The conditions of the Rolle's theorem are satisfied.

$\therefore$ We may have to find some $c \in [-1, 1]$ such that $f'(c) = 0$

Now $f(x) = x^2$ $\therefore f'(x) = 2x.$ $\therefore f'(c) = 2c.$

$\therefore f'(c) = 0 \Rightarrow 2c = 0$ $\therefore c = 0$ and lies in $[-1, 1]$

$\therefore$ Rolle's Theorem is verified.

2) Let $f(x) = x^2, \quad x \in [1, 3]$

$f(x)$ is polynomial in x. $\therefore$ $f(x)$ is continuous and differentiable everywhere on its domain. i.e. (i) f is continuous on [1, 3] and (ii) f is differentiable in (1, 3).

But we have $f(1) = 1$ and $f(3) = 9$ which are not equal.

$\therefore$ The values of f at the end points are not equal i.e. $f(1) \neq f(3)$

$\therefore$ The function x^2 in (1, 3) do not satisfy all the conditions of Rolle's Theorem.

Example 2: Verify Rolle's Theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

Solution: Given $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

(i) $f(x)$ is continuous in $[-3, 0]$ since it is a product of continuous functions.

(ii) $f'(x) = (2x+3)e^{-x/2} + (x^2+3x)\left(-\frac{1}{2}\right)e^{-x/2} = e^{-x/2}\left[2x+3-\frac{x^2}{2}-\frac{3x}{2}\right]$

$= e^{-x/2}\left[-\frac{x^2}{2}+\frac{x}{2}+3\right]$ exists in (-3, 0)

(iii) $f'(-3) = f(0) = 0.$

All conditions of Rolle's Theorem are satisfied. $\therefore$ There exists $c \in (-3, 0)$ such that $f'(c) = 0 \quad \Rightarrow e^{-c/2}\left[-\frac{c^2}{2}+\frac{c}{2}+3\right] = 0$

$\Rightarrow -c^2 + c + 6 = 0 \quad \Rightarrow c^2 - c - 6 = 0$

$\therefore c = 3, -2$

$\because 3 \notin -3, 0 \qquad \therefore c \neq 3, \quad \Rightarrow c = -2 \in -3, 0$

Hence Rolle's theorem is verified and $c = -2$ is the required value.

Example 3: Verify Rolle's Theorem for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a, b]$; $a, b > 0$

Solution: $f(x)$ is continuous in (a, b) and $f(x) = \log(x^2+ab) - \log x - \log(a+b)$

$\therefore f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$ exists, since it is not indeterminate or infinite.

Also $f(a) = f(b) = 0$ $\therefore$ All conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists $c \in (a, b)$ such that $f'(c) = 0$

$\therefore \frac{c^2-ab}{c(c^2+ab)} = 0$ (i.e.) $c^2 - ab = 0 \quad \therefore c = \sqrt{ab}$, which lies in ($a$, b).

Example 4: Verify Rolle's Theorem for $f(x) = e^{-x}(\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$.

Solution: Since e^{-x}, sin$x$, cos$x$ are continuous and differentiable functions, the given functions is also continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Also, $f(\pi/4) = e^{-\pi/4}(\sin\pi/4 - \cos\pi/4) = 0$

$$f(5\pi/4) = e^{-5\pi/4}\left(\sin 5\pi/4 - \cos 5\pi/4\right) = 0$$

$$\therefore f(\pi/4) = f(5\pi/4) = 0$$

Hence, Rolle's Theorem is applicable.

Now, $f'(x) = -e^{-x}\left(\sin x - \cos x\right) + e^{-x}\left(\cos x + \sin x\right) = 2e^{-x}\cos x$

$$f'(c) = 2e^{-c}\cos c = 0 \qquad \therefore c = \pi/2, \text{ which lies in } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Example 5: Verify Rolle's theorem for $f(x) = \sin^2 x,\ 0 \le x \le \pi$.

Solution: We have $f(x) = \sin^2 x,\ 0 \le x \le \pi$

Since $\sin x$ is continuous and differentiable on $[0, \pi]$, $\sin^2 x$ is also continuous and differentiable in the given domain. Now $f(0) = f(\pi) = 0$

$\therefore$ all the conditions of Rolle's Theorem are satisfied.

$\therefore$ The derivative of $f(x)$ should vanish for at least one point $c \in (0, \pi)$ such that $f'(c) = 0$ Now, $f'(x) = 2\sin x \cos x = \sin 2x$.

$\therefore f'(c) = \sin 2c. \Rightarrow \quad f'(c) = 0 \Rightarrow \sin 2c = 0 \Rightarrow \quad 2c = 0, \pi, 2\pi, 3\pi, \ldots$

$\therefore c = 0, \frac{\pi}{2}, \pi, \ldots$

Since $c = \frac{\pi}{2}$ lies in $(0, \pi)$, it is the required value. Hence Rolle's theorem is verified.

Example 6: If $f(x), \varphi(x), \phi(x)$ are differentiable in (a, b), show that there exists a value c in (a, b) such that $\begin{vmatrix} f(a) & \phi(a) & \varphi(a) \\ f(b) & \phi(b) & \varphi(b) \\ f'(c) & \phi'(c) & \varphi'(c) \end{vmatrix} = 0$

Solution: Consider the function $F(x)$ defined by, $F(x) = \begin{vmatrix} f(a) & \phi(a) & \varphi(a) \\ f(b) & \phi(b) & \varphi(b) \\ f(x) & \phi(x) & \varphi(x) \end{vmatrix}$

Since $f(x), \phi(x), \varphi(x)$ are differentiable in (a, b), $F(x)$ is also differentiable in (a, b). Further, $F(a) = 0$ and $F(b) = 0$ since in each case, two rows of the above determinant becomes identical. $\quad \therefore F(a) = F(b)$

Hence by Rolle's Theorem, there is a value $c \in (a, b)$ such that $F'(c) = 0$

i.e $\begin{vmatrix} f(a) & \phi(a) & \varphi(a) \\ f(b) & \phi(b) & \varphi(b) \\ f'(c) & \phi'(c) & \varphi'(c) \end{vmatrix} = 0$

Example 7: If $f(x) = x(x+1)(x+2)(x+3)$ then show that $f(x)$ has three real roots in $[-3, 0]$.

Solution: We apply Rolle's Theorem to $f(x)$ in three intervals $[-1,0]$, $[-2,-1]$, $[-3,-2]$

We observe that

(i) $f(x)$ is continuous in all the intervals since it is a polynomial in x.

(ii) $f(x)$ is differentiable in all the intervals $\therefore$ polynomial in x.

(iii) $f(-3) = f(-2) = f(-1) = f(0) = 0.$

Hence Rolle's Theorem is applicable in all each interval such that $f'(c) = 0$

$\therefore f(x)$ has three real roots.

Example 8: Prove that between any two real roots of the equation, $e^x \sin x = 1$ there is at least one roots of $e^x \cos x + 1 = 0$.

Solution: Let *a* and *b* be two real roots of the equation $e^x \sin x = 1$

(i.e.) of $\sin x = e^{-x}$ (i.e.) of $e^x - \sin x = 0$

Let $f(x) = e^{-x} - \sin x$, which is continuous and differentiable.

Also, $f(a) = f(b) = 0$. Since *a* and *b* are roots of $f(x)$.

$\therefore$ By Rolle's Theorem there is at least one real value *c* between *a* and *b* such that $f'(c) = 0$

Now, $f'(x) = -e^{-x} - \cos x$

$\therefore f'(c) = -e^{-c} - \cos c$

$f'(c) = 0 \Rightarrow -e^{-c} - \cos c = 0$

$e^{-c} + \cos c = 0$

$\therefore e^c \cos c + 1 = 0$

$\therefore$ *c* is a root of $e^x \cos x + 1 = 0$ lying between *a* and *b*.

Example 9: Us Rolle's Theorem to prove that the equation $ax^2 + bx = \frac{a}{3} + \frac{b}{2}$ has a root between 0 and 1.

Solution: Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} - \left(\frac{a}{3} + \frac{b}{2}\right)x$ which is obtained by integrating the given equation.

Here $f(x)$ is continuous in $[0,1]$ and differentiable in (0, 1) and $f(0) = f(1) = 0$

By Rolle's Theorem there is a value $c \in (0,1)$ such that $f'(c) = 0$

Now, $f'(x) = ax^2 + bx - \left(\frac{a}{3} + \frac{b}{2}\right)$ and this is zero at *x* = *c* which means the equation, $ax^2 + bx = \left(\frac{a}{3} + \frac{b}{2}\right)$ has a root between 0 and 1.

Example 10: Show that the equation $x^3 + x - 1 = 0$ where $x \in \mathbb{R}$ has exactly one real root.

Solution: Let $f(x) = x^3 + x - 1, \quad x \in \mathbb{R}$

$$f(0) = -1 < 0 \quad \text{and} \quad f(1) = 1 > 0$$

Since $f(x)$ is a polynomial, it is continuous.

Thus, using Intermediate value theorem, we get, there is a number *c* between 0 and 1 such that $f(c) = 0$

Thus the given equation has a root.

Now, let if possible $f(x)$ have two roots, say *a* and *b*. Then $f(a) = f(b) = 0$.

Since $f(x)$ represents a polynomial, it is differentiable on (*a*, *b*) and continuous on $[a,b]$

Thus by Rolle's Theorem there exists a member *c* between *a* and *b* such that $f'(c) = 0$

But $f'(x) = 3x^2 + 1, \quad x \in \mathbb{R}$

$$\therefore f'(x) \geq 1, \qquad \forall x \in \mathbb{R}$$

Hence $f'(x) \neq 0$ for any *x*, which is a contradiction.

Thus, the equation $f(x) = 0$ cannot have two real roots.

$\therefore$ The equation $x^3 + x - 1 = 0, \;\; x \in \mathbb{R}$ has exactly one root.

Check Your Progress

1. Verify the validity of the conditions and the conclusion of Rolle's Theorem for the function f defined on the intervals as given below:

a) $x^2 - 3x + 2$ on $[1,2]$

b) $\log\left[\dfrac{x^2+6}{5x}\right]$ on $[2,3]$

c) $e^{-x}\sin x$ on $[0,\pi]$

d) $e^x(\sin x - \cos x)$ on $[\pi/4, 5\pi/4]$

e) $x^2(1-x^2)$ on $[0,1]$

f) $(x-1)(x-3)e^{-x}$ in $[1,3]$

2. Prove that the equation $2x^3 - 3x^2 - x + 1 = 0$ has at least one root between 1 and 2.

3. Test whether Rolle's Theorem holds true for $f(x) = |x|$ in $[-1,1]$

4. Verify Roll's Theorem for the function $f(x) = \dfrac{\sin x}{e^x}$ in $[0,\pi]$

5. Show that $x^3 + 4x + 1 = 0$ has exactly one real solution.

Ans:(1) $c = 3/2$ (2) $c = \pi/4$ (3) $c = \pi$ (4) $c = \dfrac{1}{\sqrt{2}}$ (5) $c = 3 - 2\sqrt{2}$

12.1.2 LAGRANGE'S MEAN VALUE THEOREM

Theorem 6.1 : *If $y = f(x)$ is a real valued function defined on $[a, b]$, such that, (i) $f(x)$ is continuous on a closed interval $[a, b]$, (ii) $f(x)$ is differentiable in (a, b) then there exists at least one point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

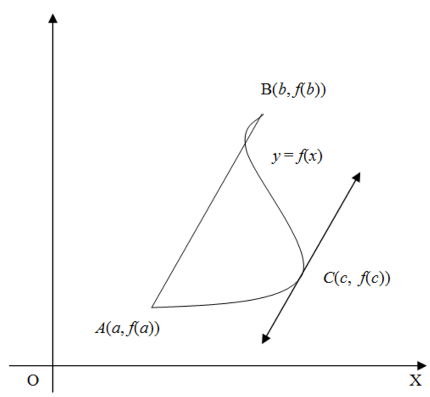
12.1.3 Another form of Lagrange's Mean value Theorem:

If (i) $f(x)$ is continuous in the closed interval $[a, a+h]$, (ii) $f(x)$ is differentiable in the open interval (a, a + h) then there exists at least one number θ in (0, 1) such that, $f(a+h) = f(a) + hf'(a + \theta h)$

12.1.4 Geometrical Interpretation of the Langrange's Mean Value Theorem:

Let $A(a, f(a))$ and $B(b, f(b))$ be two points on the curve $y = f(x)$. The slope *m* of the line *AB* is given by, *m* $= \dfrac{f(b) - f(a)}{b - a}$

Also, $f'(c)$ is the slope of the tangent at the point *C* $(c, f(c))$. Lagrange's Mean Value Theorem says that there exists at least one point $C(c, f(c))$ on the graph where the slope of the tangent line is same as the slope of line *AB*. (i.e.) *C* is a point on the graph where the tangent is parallel to the chord joining the extremities of the curve.



Physical Significance:

We note that $f(b) - f(a)$ is the change in the function $f(x)$ as *x* changes from *a* to b, so that $\dfrac{f(b) - f(a)}{b - a}$ is the change rate of change of the function $f(x)$ over $[a, b]$. Also $f'(c)$ is the actual rate of change of the function for $x = c$.

Thus the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval.

12.1.5 Some Important Deductions from the Mean Value Theorem:

Definitions:-

(i) **Monotonically increasing function**:

Let $f(x)$ be defined in $[a,b]$. Let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$. If $f(x_1) < f(x_2)$ then $f(x)$ is said to be a monotonically increasing function.

(ii) **Monotonically decreasing function**:

Let $f(x)$ be defined in $[a,b]$. Let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$. If $f(x_1) > f(x_2)$ then $f(x)$ is said to be a monotonically decreasing function.

Note:

(i) If $f(x)$ is monotonically increasing $(\uparrow)$ in $[a,b]$ then we can write $f(a) < f(x) < f(b)$ for all $x \in (a,b)$. $f(a)$ is its minimum value and $f(b)$ is its maximum value.

(ii) If $f(x)$ is monotonically decreasing $(\downarrow)$ function in $[a,b]$ then we can write, $f(a) > f(x) > f(b)$ for all $x \in (a,b)$. $f(a)$ is its maximum value and $f(b)$ is its minimum value.

(iii) Let $f(x)$ be differentiable in an interval (*a*, *b*). Let $x_1, x_2 \in (a,b)$ and $x_1 < x_2$ then applying Lagrange's Mean Value Theorem to $[x_1, x_2]$, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

or $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$ (*)

(1) Let $f'(x) > 0$ for every value of *x* in (a,b) then from equation (*)

$f(x_2) - f(x_1) > 0$ for $(x_2 - x_1)$ and $f'(c)$ both are positive i.e. $f(x_2) > f(x_1)$

We have thus proved: A function whose derivative is positive for every value of *x* in an interval is a monotonically increasing function of x in that interval.

(2) Let $f'(x) < 0$ for every value of *x* in (*a*, *b*) from equation we have,

$$f(x_2) - f(x_1) < 0 \qquad \therefore f(x_2) < f(x_1)$$

for $x_2 - x_1$ is positive and $f'(c)$ negative.

Hence $f(x)$ is a decreasing function of *x*.

We have thus proved: A function whose derivative is negative for every value of x in an interval is a monotonically decreasing function to *x* in that interval.

Example 11: Verify mean value theorem for $f(x) = \log x$ on $[1,e]$

Solution: The given function is $f(x) = \log x$ on $[1,e]$

We know that $f(x) = \log x$ is continuous on $[1,e]$ and differentiable on (1, e).

Thus all the conditions of Lagrange's mean value theorem are satisfied.

$\therefore\ \exists c \in (1,e)$ such that $\dfrac{f(e) - f(1)}{e-1} = f'(c)$

$$\therefore \frac{\log e - \log 1}{e-1} = f'(c)$$

Since $\log e = 1$, $\log 1 = 0$ and $f'(x) = \frac{1}{x}$ we get $\frac{1}{e-1} = \frac{1}{c}$

$\therefore c = e-1$ which lies in the interval (1, 2) and hence in (1, *e*), since 2 < *e* < 3.

Example 12: Separate the interval in which the polynomial $2x^3 - 15x^2 + 36x + 10$ is increasing or decreasing.

Solution: We have, $f(x) = 2x^3 - 15x^2 + 36x + 10$

$$\therefore f'(x) = 6x^2 - 30x + 36$$

(i) $f(x)$ is an increasing function if $f'(x) > 0$

i.e. $6x^2 - 30x + 36 > 0$. i.e. $x^2 - 5x + 6 > 0$

But $x^2 - 5x + 6 = (x-3)(x-2)$

$\therefore x^2 - 5x + 6 > 0$ if $(x - 3 > 0$ and $x - 2 > 0)$ or $(x - 3 < 0$ and $x - 2 < 0)$

i.e. if $(x > 3$ and $x > 2)$ or $(x < 3$ and $x < 2)$

i.e. if $x > 3$ or $x < 2$

Hence $f(x)$ is an increasing function if *x* lies in $(-\infty, 2)$ or $(3, \infty)$

(ii) $f(x)$ is a decreasing function if $f'(x) < 0$.

i.e. $6x^2 - 30x + 36 < 0$

i.e.if $x^2 - 5x + 6 < 0$

But $x^2 - 5x + 6 = (x-3)(x-2)$

$\therefore x^2 - 5x + 6 < 0$ if $(x - 3 > 0$ and $x - 2 < 0)$ or $(x - 3 < 0$ and $x - 2 > 0)$

i.e. if $(x > 3$ and $x < 2)$ or $(x < 3$ and $x > 2)$

i.e. if $x < 3$ and $x > 2$ since $x > 3$ and $x < 2$ is impossible.

i.e. if $2 < x < 3 \Rightarrow f(x)$ is decreasing in (2, 3)

Thus $f(x)$ is increasing in $(-\infty, 2)$ and $(3, \infty)$ and $f(x)$ is decreasing in (2, 3).

Example 13: Find the interval in which $f(x) = x + \frac{1}{x}$ is increasing or decreasing.

Solution: We have $f(x) = x + \frac{1}{x}$ $\quad \therefore f'(x) = 1 - \frac{1}{x^2}$

$$\therefore f'(x) = \frac{x^2 - 1}{x^2}$$

(i) $f(x)$ is an increasing function if $f'(x) > 0$

i.e. if $\frac{x^2 - 1}{x^2} > 0$.

i.e. if $x^2 - 1 > 0$.

i.e. if $x^2 > 1 \Rightarrow x > 1$ or $x < -1$

Hence $f(x)$ is increasing in the interval $(-\infty,1)$ and $(1,\infty)$

(ii) $f(x)$ is a decreasing function if $f'\ x\ < 0.$

i.e. $\frac{x^2-1}{x^2} < 0$ $\therefore x^2 - 1 < 0$ i.e if $x^2 < 1$.

i.e. if $|x| < 1$ $\Rightarrow$ $-1 < x < 1$

Hence $f(x)$ is decreasing in (-1, 1).

Example 14: Show that if $x > 0$, $x - \frac{x^2}{2} < \log\ 1+x\ < x - \frac{x^2}{2\ 1+x}$ for $x > 0$.

Solution: Let us assume, $f\ x\ = \log\ 1+x\ - x + \frac{x^2}{2}$

$$\therefore f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x}.$$

$\therefore f'(x) > 0$ for all $x > 0$ except at $x = 0.$ and $f(0) = 0.$

$\therefore f(x)$ is an increasing function in $(0,\infty)$

$\therefore f(x)$ increasing from 0 and hence $f(x) > 0$.

$$\log(1+x) < x - \frac{x^2}{2}, \text{ for } x > 0$$

… (i)

Consider,

$$f(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$$

$$f'(x) = 1 - \frac{2x - x^2}{2(1+x)^2} - \frac{1}{1+x} = \frac{x^2}{2(1+x)^2}$$

$\therefore f'(x) > 0$ for $x > 0$ except at $x = 0$ when it is zero.

$f(x)$ is an increasing function in $(0,\infty)$

$f(x)$ increasing from 0 and hence $f(x) > 0.$

$$\therefore x - \frac{x^2}{2(1+x)^2} > \log(1+x) \text{ for } x > 0.$$

… (ii)

From (i) and (ii), $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)^2}$ for $x > 0.$

Example 15: Show that $\left|\tan^{-1} x - \tan^{-1} y\right| < \left|x - y\right|$

Solution: Let $f\left(x\right) = \tan^{-1}\left(x\right)$

$\therefore$ By Lagrange's Theorem,

$$\frac{f(b) - f(a)}{b-a} = f^{'}(c)$$

$$\therefore \frac{\tan^{-1}\left(x\right) - \tan^{-1}\left(y\right)}{x-y} = \frac{1}{1+c^2} \text{ for } -\pi/2 < x < c < y < \pi/2$$

But, $\frac{1}{1+c^2} < 1$ ($\because c^2$ is positive)

$$\therefore \left|\frac{\tan^{-1} x - \tan^{-1} y}{x-y}\right| < 1$$

$$\therefore \left|\tan^{-1} x - \tan^{-1} y\right| < \left|x - y\right|$$

Example 16: Show that $\log_{10}\left(x+1\right) = \frac{x \log_{10} e}{1+\theta x}$ where $x > 0$ and $0 < \theta < 1$

Solution: Let $f\left(x\right) = \log_{10}\left(x+1\right)$

$\therefore$ By second form of Lagrange's MVT

For $\left[0, x\right]$ we have,

$$f \; a+h \; = f \; a \; + hf' \; a + \theta h$$

putting, a = 0 and h = x.

$$f\left(x\right) = f\left(0\right) + xf'\left(\theta x\right)$$

$$= 0 + xf'\left(\theta x\right) = xf'\left(\theta x\right)$$

But, $f'\left(x\right) = \frac{1}{\left(x+1\right)\log_e 10}$

$$\therefore f'\left(\theta x\right) = \frac{1}{\left(1+\theta x\right)\log_e 10} = \frac{\log_{10} e}{1+\theta x}$$

But, $f \; x \; = xf' \; \theta x$

$$\therefore \frac{f\left(x\right)}{x} = f^{'}\left(\theta x\right) = \frac{\log_{10} e}{1+\theta x}$$

$$\therefore \log\left(x+1\right) = \frac{x \log_{10} e}{1+\theta x}$$

Example 17: Applying Lagrange's M.V.T. to e^x, determine θ in terms of a and h. Hence deduce that, $0 < \frac{1}{x}\log\left(\frac{e^x - x}{x}\right) < 1$.

Solution: Let $f\left(x\right) = e^x$ $\quad \therefore f'\left(x\right) = e^x$

Now, $f(a+h) = f(a) + hf'(a+\theta h)$

$$\therefore e^{a+h} - e^{a} = he^{(a+\theta h)}$$

$$\therefore e^{a}\left(e^{h}-1\right) = he^{a}.e^{\theta h}$$

$$e^{\theta h} = \frac{e^{h}-1}{h}$$

$$\therefore \theta h = \log\left(\frac{e^{h}-1}{h}\right)$$

$$\therefore \theta = \frac{1}{h}\log\left(\frac{e^{h}-1}{h}\right)$$

But, $0 < \theta < 1 \qquad \therefore 0 < \frac{1}{h}\log\left(\frac{e^{h}-1}{h}\right) < 1$

Now by substituting $h = x$ in the above equation, we get,

$$\therefore 0 < \frac{1}{x}\log\left(\frac{e^{x}-1}{x}\right) < 1$$

Example 18: Show that, $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$

Hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \pi/4 + \frac{1}{6}$

Solution: Let $f(x) = \tan^{-1}(x)$ in $[a,b]$

$$\therefore f'(x) = \frac{1}{1+x^2}$$

$\therefore$ By Lagrange's M. V. T.

$f'(c) = \frac{f(b)-f(a)}{b-a}$ where $c \in (a,b)$

$$\therefore \frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} \qquad (1)$$

Since $a < c < b, \qquad a^2 < c^2 < b^2$

$$\therefore 1+a^2 < 1+c^2 < 1+b^2$$

$$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \qquad (2)$$

From (1) and (2)

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\therefore \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \qquad (3)$$

For the second part;

Since $\tan^{-1} = \pi/4$ we put $a = 1$ and $b = \frac{4}{3}$ in (3)

$$\therefore \frac{\frac{4}{3}-1}{1+\left(\frac{16}{9}\right)} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$$

$$\therefore \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}.$$

Example 19: Prove that, $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$ for $0 < a < b$

Hence deduce that $\frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$

Solution: Let $f(x) = \log x$ in $[a,b]$

Since $f(x)$ is (i) continuous in $[a,b]$ and (ii) differentiable in (a, b)

by Lagrange's M. V. T. $\exists c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

But $f(x) = \log x$

$$\therefore f'(x) = \frac{1}{x} \qquad \therefore f'(c) = \frac{1}{c}$$

$$\therefore \frac{\log b - \log a}{b-a} = \frac{1}{c}$$

(1)

But $a < c < b$, $\quad \frac{1}{a} < \frac{1}{c} < \frac{1}{b}$ (2)

From (1) and (2) we get,

$$\frac{1}{b} < \frac{\log b - \log a}{b-a} < \frac{1}{a} \qquad \Rightarrow \frac{b-a}{b} < \log b - \log a < \frac{b-a}{a}$$

$$\therefore \frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$$

For the second part $a = 3$, $b = 4$.

$$\therefore \frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$$

Check Your Progress

1. Examine the validity of the conditions and the conclusions of LMVT for the functions given below:

(i) e^x on $[0,1]$ $\qquad [Ans: c = \log(e-1)]$

(ii) $\sqrt{x^2-4}$ on $[2,3]$ $\qquad [Ans: c = \sqrt{5}]$

(iii) $x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$ $\qquad \left[Ans: c = \sqrt{3/2}\right]$

(iv) $\frac{1}{x}$ on $[-1,1]$ $\qquad [Ans: Not\ applicable]$

2. Apply LMVT to the function Log x in $\left[a, a+h\right]$ and determine θ in terms of and h. Hence deduce that: $0 < \frac{1}{\log\left(1+x\right)} - \frac{1}{x} < 1$.

3. Applying LMVT show that:

(i) $\frac{1}{1+x^2} < \frac{\tan^{-1} x}{x} < 1$ for $x > 0$ (ii) $1 < \frac{\sin^{-1} x}{x} < \frac{1}{\sqrt{1-x^2}}$ for $0 \le x < 1$

(iii) $\frac{1}{x} < \frac{1}{\log\left(1+x\right)} < \frac{x+1}{x}, \quad x > 0.$

4. Prove that, $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}, \quad 0 < a < b < \pi/2$

Hence deduce that,

(i) $\pi/b + \frac{\sqrt{3}}{15} < \sin^{-1}\left(3/5\right) < \pi/b + \frac{1}{8}$ (ii) $\frac{\pi}{b} - \frac{1}{2\sqrt{3}} < \sin^{-1}\left(1/4\right) < \pi/b - \frac{1}{\sqrt{15}}$

5. Separate the intervals in which the following polynomials are increasing or decreasing. (i) $x^3 - 3x^2 - 24x - 31$ (ii) $x^3 - 6x^2 - 36x + 7$

[$Ans: (i)$Increasing $-\infty, -2$, $4, \infty$;Decreasing $-2, 4$

(ii)Increasing $-\infty, -2$ and $6, \infty$;Decreasing $-2, 6$]

6. Show that, $x - 1 > \log x > \frac{x-1}{x}$ for $1 - x$.

7. If $f\left(x\right) = x\sin x + \cos x + \cos^2 x$ then show that, $2 > f\left(x\right) > \pi/2$

12.2 Cauchy's Mean Value Theorem:

If functions f and g are (i) continuous in a closed interval [a, b], (ii) differentiable in the open interval (a, b) and (iii) $f'\ x \neq 0$ for any point of the open interval (a, b) then for some $c \in \left(a, b\right)$, $f'\ c\left[g\ b - g\ a\right] = g'\ c\left[f\ b - f\ a\right]$

i.e. $\frac{g'\ c}{f'\ c} = \frac{g\ b - g\ a}{f\ b - f\ a} \quad a < c < b.$

12.2.1 Another form of Cauchy's Mean Value Theorem:

If two function $f\left(x\right)$ and $g\left(x\right)$ are derivable in a closed interval [a, $a + h$] and $f'\ x \neq 0$ for any x in (a, $a + h$) then there exists at least one number $\theta \in \left(0, 1\right)$ such that, $\frac{g\ a+h - g\ a}{f\ a+h - f\ a} = \frac{g'\ a+\theta h}{f'\ a+\theta h}, \qquad 0 < \theta < 1$

The equivalence of the two statements can be shown as in case of Lagrange's mean value theorem.

Remark:

(i) Taking $f(x) = x$, we can derive Lagrange's mean value theorem. In other words, we may easily see that Lagrange's theorem is only a particular case of Cauchy mean value theorem.

(ii) Usefulness of this theorem depends on the fact that f' and g' are considered at the same point c. If we apply LMVT to '*f*' and '*g*' separately then $f(b) - f(a) = (b-a)f'(c_1), \quad g(b) - g(a) = (b-a)g'(c_2)$ for some $c_1, c_2 \in (a,b)$

12.2.2 Geometrical Application of Cauchy's Mean Value Theorem:

Geometrically, we consider a curve whose paramedic equations are $x = g(t)$, $y = f(t), \quad a \le t \le b$. Then, slope of the curve at any point is, $\dfrac{dy}{dx} = \dfrac{f'\ t}{g'\ t}$

Also the slope of the chord joining the end points $A[g(a), f(a)]$ and $B[g(b), f(b)]$ is given by, $\dfrac{f(b) - f(a)}{g(b) - g(a)}$

Thus under the assumption of Cauchy mean value theorem. If $x_0 \in (a,b)$ such that the tangent to the curve at $[g(x_0), f(x_0)]$ is parallel to the chord AB.

Example 20: Verify Cauchy's MVT for the function x^2 and x^3 in the interval [1, 2].

Solution: Let $f(x) = x^2$ and let $g(x) = x^3$.

As $f(x)$ and $g(x)$ are polynomials (*i*) they are continuous on [1, 2], (*ii*) they are differentiable on (1, 2) and (*iii*) $g'\ x \ne 0$ for any value in (1, 2)

$\therefore$ Cauchy's mean value theorem can be applied. $\therefore$ If $c \in\ 1,2$ such that,

$$\frac{f'\ c}{g'\ c} = \frac{f\ 2\ - f\ 1}{g\ 2\ - g\ 1}$$

$$\frac{2c^2}{3c^2} = \frac{2^2 - 1^2}{2^3 - 1^3} = \frac{4-1}{8-1} = 3/7 \qquad \Rightarrow \qquad \frac{2}{3c} = \frac{3}{7}$$

$$\Rightarrow 9c = 14 \qquad \therefore c = \frac{14}{9} \in\ 1,\ 2$$

$\therefore$ Cauchy mean value theorem is verified.

Example 21: Using CMVT show that $\dfrac{\sin b - \sin a}{\cos a - \cos b} = \cot c, \quad a < c < b,$ $a > 0, b > 0$

Solution: Let $f(x) = \sin x$ and $g(x) = \cos x$.

Here, $f(x)$ and $g(x)$ are continuous on [*a*, *b*] and differentiable on (*a*, *b*) and for any c in (*a*, *b*), thus CMVT can be applied.

$\therefore c \in (a,b)$ such that, $\dfrac{f'\ c}{g'\ c} = \dfrac{f\ b\ - f\ a}{g\ b\ - g\ a}$

$$\therefore \frac{-\cos c}{\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a} \qquad \Rightarrow \qquad \cot c = \frac{\sin b - \sin a}{\cos a - \cos b}$$

Example 22: If in CMVT we write $f(x) = e^{x}$ and $g(x) = e^{-x}$ show that *c* is the arithmetic mean between *a* and *b*.

Solution: Now $f(x) = e^{x}$ and $g(x) = e^{-x}$

If can be proved that function $f(x)$ and $g(x)$ are continuous on any closed interval [*a*, *b*] and differentiable in (*a*, *b*). Also $g'(x) \neq 0$ and $x \in (a,b)$

Then CMVT can be applied. $\therefore \exists\ c \in (a,b)$ such that, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

Now $\dfrac{f'(c)}{g'(c)} = \dfrac{e^{c}}{-e^{-c}} = -e^{2c}$ and $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{e^{b} - e^{a}}{e^{-b} - e^{-a}} = -e^{a+b}$ where $c \in (a,b)$

$\therefore -e^{2c} = -e^{a+b} \qquad \Rightarrow a + b = 2c$

$\therefore c = \dfrac{a+b}{2} \in (a,b)$

Thus, *c* is the arithmetic mean between *a* and *b*.

Example 23: Using CMVT prove that there exists a number *c* such that $0 < a < c < b$ and $f(b) - f(a) = cf'(c) \log \left(b/a\right)$. By putting $f(x) = x^{1/n}$ deduce that

$$\lim_{n\to\infty} n\left(b^{1/n} - 1\right) = \log b.$$

Solution: Let $f(x)$ be a continuous and differentiable function and $g(x) = \log x$.

Then $f(x)$ and $g(x)$ satisfy the condition of continuity and differentiability of CMVT. Hence $\exists\ c \in (a,b)$ such that, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\therefore \frac{f'(c)}{1/c} = \frac{f(b) - f(a)}{\log b - \log a} \qquad \Rightarrow \qquad f(b) - f(a) = cf'(c) \log \left(b/a\right)$$

If $f(x) = x^{1/n}$ and $g(x) = \log x$ then by putting *a* = 1 we get in the interval (1, *b*)

$$\frac{b^{1/n} - 1}{\log b - \log 1} = \frac{(1/n)c^{\frac{1}{n}-1}}{1/c} \quad \text{where } 1 < c < b$$

$$\therefore n\left(b^{1/n} - 1\right) = (\log b)\, c^{1/n}.$$

$$\lim_{n\to\infty} n\left(b^{1/n}-1\right)=\log b \quad [c^{1/n}\to 1 \text{ as } n\to\infty]$$

Example 24: If $1<a<b$, show that there exists c satisfying $a<c<b$ such that

$$\log b/a=\frac{b^2-a^2}{2c^2}$$

Solution: We have to prove that, $\frac{\log b-\log a}{b^2-a^2}=\frac{1}{2c^2}$

This suggests us to take $f(x)=\log x$ and $g(x)=x^2$ Now, $f(x)$ and $g(x)$ are continuous on [*a*, *b*] and differentiable on (*a*, *b*) and $g'(x)\neq 0$ for any *c* in (*a*, *b*).

$\therefore$ CMVT can be applied. $\therefore\ \exists\ c\in(a,b)$ such that,

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)} \qquad \Rightarrow \frac{1/c}{2c}=\frac{\log b-\log a}{b^2-a^2}$$

$$\therefore \frac{1}{2c^2}=\frac{\log b-\log a}{b^2-a^2} \quad \Rightarrow \log b/a=\frac{b^2-a^2}{2c^2}$$

Check Your Progress

1. Find *c* of Cauchy's mean value theorem for:

(i) $f(x)=\sqrt{x},\quad g(x)=\frac{1}{\sqrt{x}},\quad x\in[a,b],\quad a>0$ **(ii)** $f(x)=\sin x$, $g(x)=\cos x$ on $[0,\pi/2]$

(iii) $f(x)=3x+2,\quad g(x)=x^2+1$ on $1\le x\le 4$. **(iv)** $f(x)=e^x$, $g(x)=e^{-x}$ on [0, 1]

(v) $f(x)=e^x,\quad g(x)=\frac{x^2}{x^2+1},\quad x\in[-1,1]$

[Ans :- (i) $\sqrt{ab}$ (ii) $\pi/4$ (iii) $5/2$ (iv) $1/2$ (v) 0.]

12.3 Summary

In this chapter we have learnt about the mean value theorems. The Rolle's theorem which is the fundamental theorem in analysis has been proved. The Lagrange's MVT and the Cauchy's MVT have also been proved. Problems based on these theorems have been done in order to understand the Mean Value theorems. In the next chapter we are going to learn about Taylor's theorem and its applications.

12.4 Unit End Exercise:

1. Verify Rolle's theorem for each of the following:

i) $f(x)=(x-1)(x-2)(x-3)$ in [-1,1]

ii) $f(x)=x(x-3)^2$ in [0,3]

iii) $f(x)=\tan 2x$ in $[0,\pi]$

iv) $f(x)=\sqrt{4-x^2}$ in [-2, 2]

2. Verify LMVT for the following functions.

$i)\ f(x) = \sqrt{x^2 - 1}$ in [-1, 1]

$ii)\ f(x) = (x-1)(x-4)(x-3)$ in [0, 7]

$iii)\ f(x) = x(x+1)^2$ in [0, 2]

3. Find 'c' of CMVT for the following:

$i)\ f(x) = x^2,\ g(x) = x^3$ in [1,2]

$ii)\ f(x) = x^2 + 2x + 4,\ g(x) = x + 3$ in [0,2]

$iii)\ f(x) = (x-1)^2 + 4,\ g(x) = x - 1$ in [0,2]

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APPROXIMATION, ERRORS AND EXTREMA 13

Unit Structure

13.1 INTRODUCTION

In the previous units we have seen how we can successively differentiate a function, partial differentiation of a function and also various mean value theorems. Differential Calculus has various applications. Some of the physical and geometrical applications we have seen before. Derivatives can also be used to find maximum and minimum values of a function in an interval. The maximum and minimum values are called extreme values of a function. The extreme values can be absolute or can be local. The first derivative test and the second derivative tests are used to determine the points of local extrema. In this chapter we are going to use the differential calculus concept to answer questions like:

(1) What is the approximate value of $\sin 1^\circ$?

(2) What is the error in calculating the area of a square, if the error in calculating the side length was 1%?

(3) What are the maximum and minimum values of a function in a given interval?

13.2 OBJECTIVES

After studying this unit you should be able to:

- compute the approximate value of a function at a given point.
- compute error, relative error and percentage error
- compute maxima and minima for a function in a given interval.

13.3 ERRORS and APPROXIMATION

Let $z = f(x, y)$ be a differentiable function. Let δx denote the error in *x* and δy denote the error in *y*. Then the corresponding **error in *z*** denoted by δz is given by:

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

The above formula can be extended to more than two variables also. For example, if $u = f(x, y, z)$ then continuing with the same notations, $\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$.

Relative error: If δz is an error in *z* then the relative error in *z* is given by: $\frac{\delta z}{z}$.

Percentage error: If δz is an error in *z* then the percentage error in *z* is given by: $\frac{\delta z}{z} \times 100$.

Approximate value: If *z* is the calculated error value and δz is the error in *z* then the approximate value is given by: $z + \delta z$.

Let us understand this with the help of some examples:

***Example* 1:** The radius of a sphere is calculated to be 12 cm with an error of 0.02 cm. Find the percentage error in calculating its volume.

Solution: Given $r = 12$ cm and $\delta r = 0.02$ cm. To find percentage error in volume of the sphere. Let *V* denote the volume of the sphere. To find $\frac{\delta V}{V} \times 100$.

Now, $V = \frac{4}{3}\pi r^3 \Rightarrow \delta V = \frac{dV}{dr}\delta r = \frac{4}{3}\pi \times 3r^2 \times \delta r$.

Thus, $\frac{\delta V}{V} \times 100 = \frac{\frac{4}{3}\pi \times 3r^2 \times \delta r}{\frac{4}{3}\pi r^3} \times 100 = 3\frac{\delta r}{r} \times 100 = 3 \times \frac{0.02}{12} \times 100 = 0.5$

The percentage error in calculating the volume is 0.5.

13.4 MAXIMA AND MINIMA

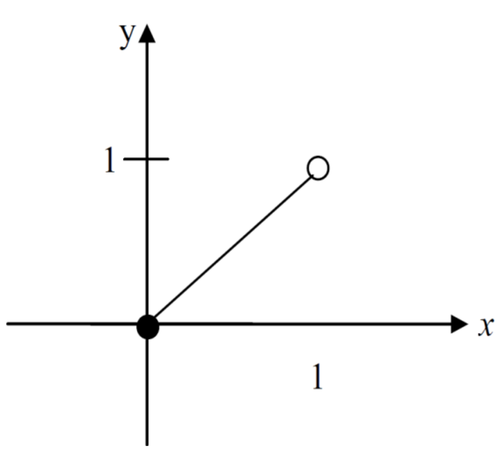
In this section, we shall study how we can use the derivative to solve problems of finding the maximum and minimum values of a function on an interval. We begin by looking at the definition of the minimum and the maximum values of a function on an interval.

Definition : Let *f* be defined on an interval *I* containing '*c*'

1. $f(c)$ is the (absolute) **minimum of *f* on *I*** if $f(c) \le f(x)$ for all *x* in *I*.
2. $f(c)$ is the (absolute) **maximum of *f* on** *I* if $f(c) \ge f(x)$ for all *x* in *I*.

The minimum and maximum of a function on an interval are called the **extreme values** or **extreme,** of the function on the interval.

Remark : A function need not have a minimum or maximum on an interval. For example $f(x) = x$ has neither a maximum nor a minimum on open interval (0,1). Similarly, $f(x) = x^3$ has neither any maximum nor any minimum value in . See figures 13.1 and 13.2.



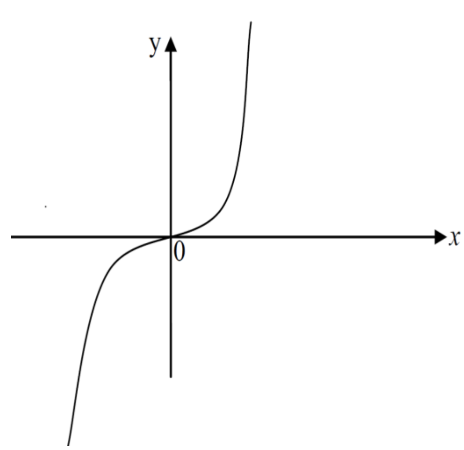


Fig. 13.1: $f(x) = x, x \in (0, 1)$

Fig. 13.2 $f(x) = x^3, x \in \mathbb{R}$

If f is a continuous function defined on a closed and bounded interval $[a,b]$, then f has both a minimum and a maximum value on the interval $[a,b]$. This is called the extreme value theorem and its proof is beyond the scope of our course.

Look at the graph of some function $f(x)$ in **Fig. 13.3.**

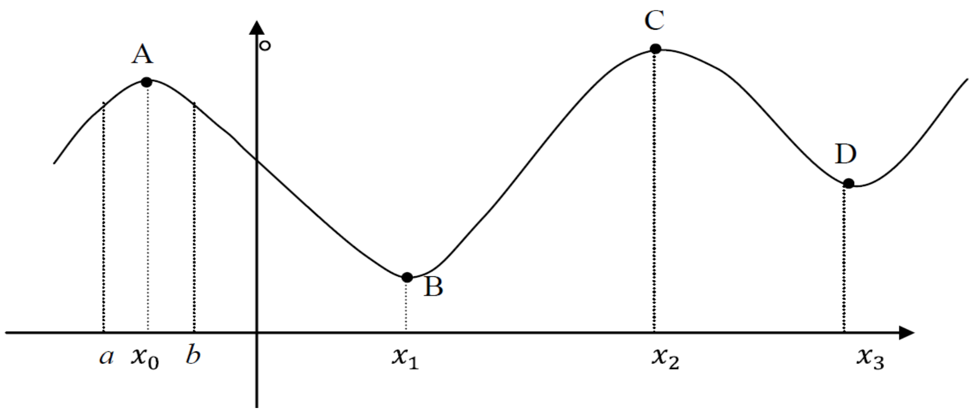


Fig. 13.3

Note that at $x = x_0$, the point A on graph is not an absolute maximum because $f(x_2) > f(x_0)$. But if we consider the interval (a,b), then f has a maximum value at $x = x_0$ in the interval (a,b). Point A is a point of local maximum of f. Similarly f has a local minimum at point B.

Definition : Suppose f is a function defined on an intervals I. f is said to have a local (relative) maximum at $c \in I$ if there is a positive number h such that for each $x \in I$ for which $c - h < x < c + h,\ x \neq c$ we have $f(x) > f(c)$.

Definition : Suppose f is a function defined on an interval I. f is said to have a local (relative) minimum at $c \in$ I if there is a positive number h such that for each $x \in$ I for which $c - h < x < c + h,\ x \neq c$ we have $f(x) < f(c)$.

Again Fig. 13.4 suggest that at a relative extreme the derivative is either zero or undefined. We call the x–values at these special points as critical numbers.

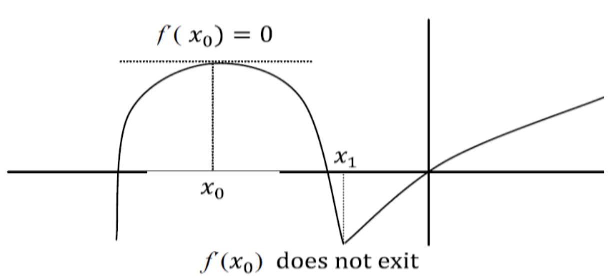


Fig 13.4

Definition :
If f is defined at c, then c is called a critical number if f if $f'(c) = 0$ or f' is not defined at c. The following theorem which we state without proof tells us that relative extreme can occur only at critical points.

Theorem: If f has a relative minimum or relative maximum at $x= c$, then c is a critical number of f.
If f is a continuous function on interval $[a,b]$, then the absolute extrema of f occur either at a critical number or at the end points a and b. By comparing the values of f at these points we can find the absolute maximum or absolute minimum of f on $[a, b]$.

Example 2 : Find the absolute maximum and minimum of the following functions in the given interval. (i) $f(x) = x^2$ on $[–3, 3]$ (ii) $f(x) = 3x^4 – 4x^3$ on $[–1, 2]$
Solution : (i) $f(x) = x^2$, $x \in [–3,3]$
Differentiating w.r.t. x., we get $f'(x) = 2x$
To obtain critical numbers we set $f'(x) = 0$. This gives $2x = 0$ or $x = 0$ which lies in the interval $(–3,3)$.
Since f' is defined for all x, we conclude that this is the only critical number of f.
Let us now evaluate f at the critical number and at the end of points of $[–3,3]$.
$f(–3) = 9$
$f(0) = 0$
$f(3) = 9$
This shows that the absolute maximum of f on $[–3,3]$ is $f(–3) = f(3) = 9$ and the absolute minimum is $f(0) = 0$
(ii) $f(x) = 3x^4– 4x^3$ $x \in [–1, 2]$ $f'(x) = 12x^3 – 12x$
To obtain critical numbers, we set $f'(x) = 0$ or $12x^3 – 12x = 0$ which implies $x = 0$ or $x = 1$.
Both these values lie in the interval $(–1, 2)$
Let us now evaluate f at the critical number and at the end points of $[–1,2]$
$f(–1) = 7$
$f(0) = 0$
$f(1) = –1$
$f(2) = 16$
This shows that the absolute maximum 16 of f occurs at $x = 2$ and the absolute minimum – 1 occurs at $x = 1$.

First Derivative Test

How do we know whether f has a local maximum or a local minimum at a critical point c ? we shall study two tests to decide whether a critical point c is a point of local maxima or local minima. We begin with the following result which is known **as first derivative test.** This result is stated without any proof.

Theorem : Let c be a critical point for f, and suppose that f is continuous at c and differentiable on some interval I containing c, except possibly at c itself. Then

(i) if f' changes from positive to negative at c, that is, if there exists some $h > 0$ such that $c - h < x < c$, implies $f'(x) > 0$ and $c < x < c + h$ implies $f'(x) < 0$, then f has a local maximum at c.

(ii) if f' changes sign from negative to positive at c, that is, if there exists some $h > 0$ such that $c - h < x < c$ implies $f'(x) < 0$ and $c < x < c + h$ implies $f'(x) > 0$ then f has a local minimum at c.

(iii) if $f'(x) > 0$ or if $f'(x) < 0$ for every x in I except $x = c$ then $f(c)$ is not a local extremum of f.

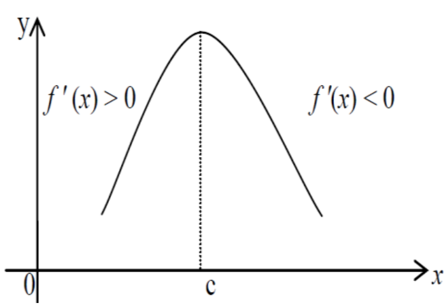


Fig 13.5

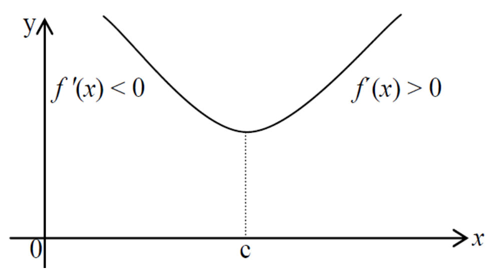


Fig 13.6

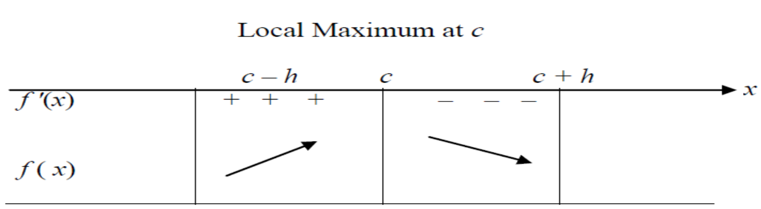
As an illustration of ideas involved, imagine a blind person riding in a car. If that person could feel the car travelling uphill then downhill, he or she would know that the car has passed through a high point of the highway. Essentially, the sign of derivative $f'(x)$ indicates whether the graph goes uphill or downhill. Therefore, without actually seeing the picture we can deduce the right conclusion in each case. We summarize the first derivative test for local maxima and minima as following:

First Derivative Test for Local Maxima and Minima

Let c be a critical number of f i.e., $f'(c) = 0$

If $f'(x)$ changes sign from positive to negative at c then $f(c)$ is a local maximum. See fig 13.7. If $f'(x)$ changes sign from negative to positive at c then $f(c)$ is a local minimum. See fig 13.8.

Note : $f'(x)$ does not change, sign at c, then $f(c)$ is neither a local maximum nor local minimum.



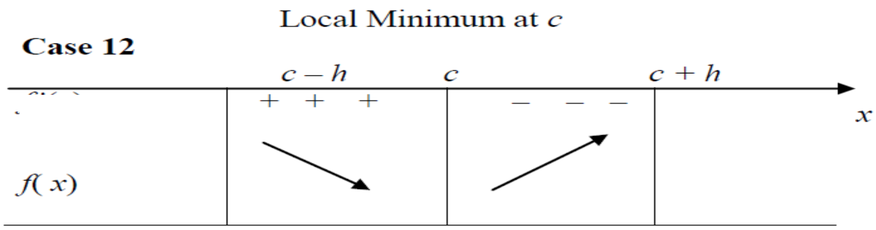


Fig 13.7

Fig 13.8

Example 3: Find the local (relative) extrema of the following functions :

(i) $f(x) = 2x^3 + 3x^2 - 12x + 7$ (ii) $f(x) = \dfrac{1}{x^2+2}$ (iii) $f(x) = x.e^x$

Solution

(i) f is continuous and differentiable on **R**, the set of real numbers. Therefore, the only critical values of f will be the solutions of the equation $f'(x) = 0$.

Now, $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$

Setting $f'(x) = 0$ we obtain $x = -2, 1$

Thus, $x = -2$ and $x = 1$ are the only critical numbers of f. Figure 13.9 shows the sign of derivative f' in three intervals. From Figure 13.9 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 1$, $f'(x) < 0$ and if $x > 1$, $f'(x) > 0$.

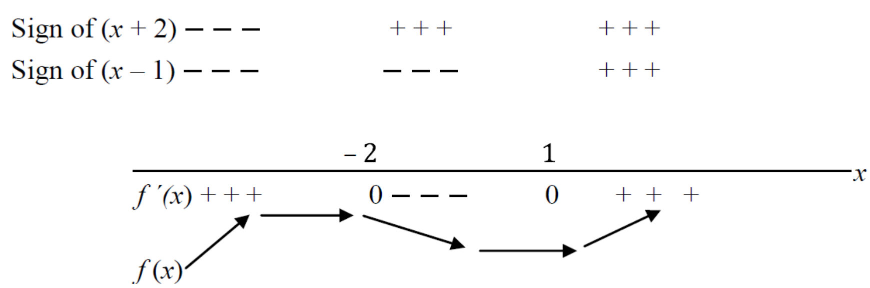


Fig 13.9

Using the first derivative test we conclude that $f(x)$ has a local maximum at $x = -2$ and $f(x)$ has local minimum at $x = 1$.

Now, $f(-2) - 2 = 2(-2)^3 + 3(-2)^2 - 12(-2) + 7 = -16 + 12 + 24 + 7 = 27$ is the value of local maximum at $x = -2$ and $f(1) = 2 + 3 - 12 + 7 = 0$ is the value of local minimum at $x = 1$.

(ii) Since $x^2 + 2$ is a polynomial and $x^2 + 2 \neq 0$ is continuous and differentiable on **R**, the set of real numbers. Therefore, the only critical values of $f(x) = \dfrac{1}{x^2+2}$ will be the solutions of the equation $f'(x) = 0$.

Setting $f'(x) = 0$ we obtain $x = 0$. Thus, $x = 0$ is the only critical number of f. Figure 13.10 shows the sign of derivative in two intervals.

Now $f'(x) = \dfrac{-2x}{(x^2+2)^2}$

Setting $f'(x) = 0$ we obtain $x = 0$. Thus, $x = 0$ is the only critical number of f. Figure 24 shows the sign of derivative in two intervals.

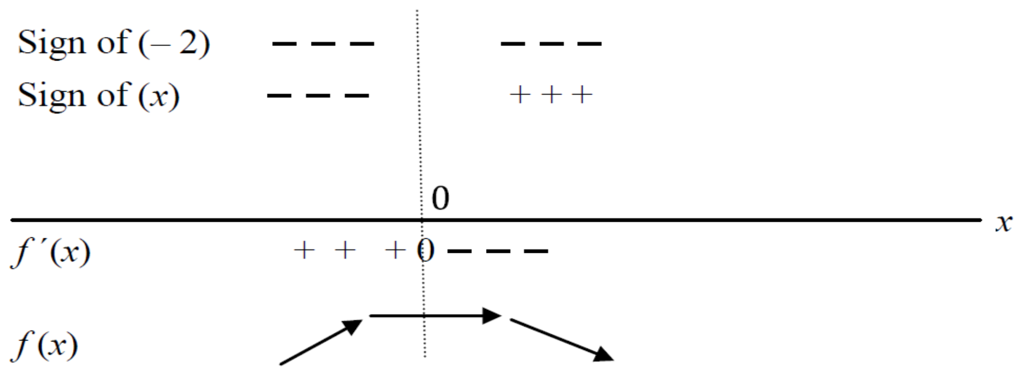


Fig. 13.10

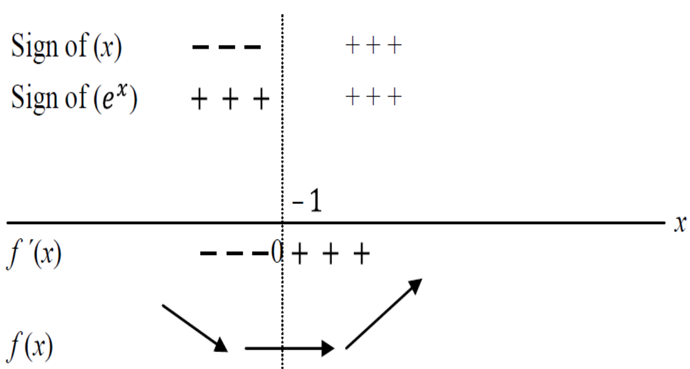
From Figure 13.10 it is clear that $f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0$. Using the first derivative test, we conclude that f(x) has a local maximum at x = 0.

Now since $f(0) = \dfrac{1}{0^2+2} = \dfrac{1}{2}$ the value of the local maximum at $x = 0$ is 1/2.

(iii) Since x and e^x are continuous and differentiable on **R**, $f(x) = xe^x$ is continuous and differentiable on **R.**
Therefore, the only critical values of f will be solutions of $f'(x) = 0$.

Now, $f'(x) = xe^x + 1e^x = (x+1)e^x$

Since $e^x > 0, \forall x \in \mathbb{R}$, $f'(x) = 0$ gives $x = -1$. Thus, $x = -1$ is the only critical number of f.
The figure below shows the sign of derivative f' in two intervals :



It is clear that $f'(x) < 0$ if $x < -1$ and $f'(x) > 0$ if $x > -1$

Using the first derivative test we conclude that $f(x)$ has a local minimum at $x = -1$ and the value of local minimum is
$f(-1) = -1/e$.

Second Derivative Test

The first derivative test is very useful for finding the local maxima and local minima of a function. But it is slightly cumbersome to apply as we have to determine the sign of f ' around the point under consideration. However, we can avoid determining the sign of derivative f ' around the point under consideration, say c, if we know the sign of second derivative f at point c. We shall call it as the second derivative test.

Theorem : (Second Derivative Test)

Let f(x) be a differentiable function on I and let c $\in$I. Let f ´ (x) be continuous at c. Then

1. *c* is a point of local maximum if both *f* '(*c*) = 0 and f (*c*) < 0.
2. *c* is a point of local minimum if both *f* '(c) = 0 and *f* (*c*) > 0.

Remark : If *f*´(*c*) = 0 and *f*´´(*c*) = 0, then the second derivative test fails. In this case, we use the first derivative test to determine whether *c* is a point of local maximum or a point of a local minimum.

We summarize the second – derivative test for local maxima and minima in the following table.

Second Derivative Test for Local Maxima and Minima

$f'(c)$	$f''(c)$	$f(c)$
0	+	Local Minimum
0	-	Local Maximum
0	0	Test Fails

We shall adopt the following step to determine local maxima and minima.

Steps to find Local Maxima and Local Minima

The function *f* is assumed to posses the second derivative on the interval *I*.

Step 1 : Find *f '* (*x*) and set it equal to 0.

Step 2 : Solve *f '* (*x*) = 0 to obtain the critical numbers of *f*. Let the solution of this equation be α, ………. We shall consider only those values of *x* which lie in *I* and which are not end points of *I*.

Step 3 : Evaluate *f ´* (α) If *f* (α) < 0, *f* (*x*) has a local maximum at *x* = α and its value if *f* (α) If *f* (α) > 0, *f* (*x*) has a local minimum at *x* = α and its value if *f* (α) If *f* (α) = 0, apply the first derivative test.

Step 4 : If the list of values in Step 2 is not exhausted, repeat step 3, with that value.

Example 4: Find the points of local maxima and minima, if any, of each of the following functions. Find also the local maximum values and local minimum values.

(i) *f* (*x*) = $x^3 - 6x^2 + 9x + 1,\ x \in \mathbb{R}$

(ii) $f(x) = x^3 - 2ax^2 + a^2x \ (a > 0), x \in \mathbb{R}$

Solution:

(i) $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

To obtain critical number of *f*, we set $f'(x) = 0$ this yields $x = 1, 3$.
Therefore, the critical number of *f* are $x = 1, 3$.
Now $f''(x) = 6x - 12 = 6(x - 2)$
We have $f''(1) = 6(1 - 2) = -6 < 0$ and $f''(3) = 6(3 - 2) = 6 > 0$.
Using the second derivative test, we see that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$. The value of local maximum at $x = 1$ is $f(1) = 1 - 6 + 9 + 1 = 5$ and the value of local minimum at $x = 3$ is
$f(3) = 3^3 - 6(3)^2 + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$.

(ii) We have $f(x) = x^3 - 2ax^2 + a^2x \ (a > 0)$

Thus, $f'(x) = 3x^2 - 4ax + a^2 = (3x - a)(x - a)$

As $f'(x)$ is defined for each $x \in \boldsymbol{R}$, to obtain critical number of *f* we set $f'(x) = 0$.
This yields $x = a/3$ or $x = a$.
Therefore, the critical numbers of *f* and $a/3$ and a. Now, $f''(a) = 6x - 4a$.

We have $f''(a/3) = 6(a/3) - 4a = -2a < 0$

and $f''(a) = 6a - 4a = 2a > 0$

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = a/3$ and a local minimum at $x = a$.

The value of local maximum at $x = a/3$ is $f(a/3) = \frac{4}{27}a^3$ and the value of local minimum at $x = a$ *is* $f(a) = 0$.

Check Your Progress

1. Find the absolute maximum and minimum of the following functions in the given intervals.
(i) $f(x) = 4 - 7x + 3$ on $[-2, 3]$

(ii) $f(x) = \frac{x^3}{x+2}$ on $[-1,1]$

2. Using first derivative test find the local maxima and minima of the following functions.

(i) $f(x) = x^3 - 12x$ (ii) $f(x) = \frac{x}{2} + \frac{2}{x}, \ x > 0$

3. Use second derivative test to find the local maxima and minima of the following functions.

(i) $f(x) = x^3 - 2x^2 + x + 1, \ x \in \mathbb{R}$ (ii) $f(x) = x + 2\sqrt{1-x}, \ x \le 1$

13.5 LET US SUM UP

The chapter is, as suggested by the title, on applications of differential calculus. In section 13.4, methods for finding out (local) maxima and minima, are discussed and explained with examples.